Problem N1. $x\left(8\sqrt{1-x}+\sqrt{1+x}\right) \leq 11\sqrt{1+x}-16\sqrt{1-x}$, $x>0$.

Substitute $y=\sqrt{\frac{1+x}{1-x}}$ or vice versa.

Problem N2. In a tetrahedron, the sums of opposite edges are equal. Prove that the sums of opposite dihedral angles are equal.

N3. The bisector of a trihedral angle is defined as the l.o.p. [locus of points] equidistant from its edges. Find a necessary and sufficient condition for two bisectors of a tetrahedron to intersect.

N4. Among tetrahedra with a given volume, find the tetrahedron with the maximum radius of the inscribed sphere.

N5. M - set of numbers of the form $A+B\sqrt{2}$, A and B - integers. Prove that in any neighborhood of any number, there is an infinite number of elements from the set M.

N6. Pt. O lies inside $\triangle$-angle ABC, such that $\angle ABC=80$, $\angle OAC=10$; $\angle OCA=30$. $BO:AC=k$. Find $\angle ABO$.

N7. Find all functions $F(x)$ such that for any x_1 and x_2 $F(x_1)-F(x_2)\leq (x_1-x_2)^2$.

N8. Solve the eq-tion: $\sqrt{107+\sqrt{107+x}}=x$

Sol. ($\sqrt{107+x}=y \implies \sqrt{107+y}=x \implies y=x \qquad \sqrt{107+x}=x$)

N9. The numbers $\alpha_1,\alpha_2,\ldots,\alpha_n$ form an arithmetic progression. Find $\displaystyle\sum_{i=1}^{n-1}\frac{1}{\cos\alpha_i\cos\alpha_{i+1}}$

N10. Given $\triangle$ ABC. Construct pt. K on side AB and pt. M on side BC such, that $AK=KM=MC$.

N11. In tetrahedron $KABC$ $AB=BC=CA$ and $\angle KAB=\angle KBC=\angle KCA$. Prove, that $KABC$ - regular pyramid.

N12. Given $\triangle$ ABC and pt. O inside it, moreover $AB=BC$, $\angle ABC=80°$; $\angle AOC=10$, $\angle OCA=30$. Find $\angle AOB$. (incorrect condition)

N13. Let R - radius of the circumscribed, and r - radius of the inscribed sphere of a tetrahedron. Prove, that $R:r\geq 3$.

N14. Is the set of pairs of integers A and B finite, such that the set of prime divisors of numbers A and B coincide; and also of the numbers $A-1$ and $B-1$?

Sol: $A = 2^n - 1, B = (2^n - 1)^2 \implies$ infinitely many numbers.

N15. All bisectors of a $\triangle$-angle are less than 1. Prove, that its area $\leq \dfrac{\sqrt{3}}{3}$.

N16. Construct a quadrilateral given its angles and diagonals.

N17. Solve the eq-tion: $\sin^7 x + \dfrac{1}{\sin^3 x} = \cos^7 x + \dfrac{1}{\cos^3 x}$.

N18. Sol. eq. $\left(1 - \dfrac{1}{8}\cos^2 x\right)^8 = 2\sin^2 x$

$$\left(1 - \frac{1}{8}\cos^2 x\right)^8 = \sin^2 x \implies y = 1 - \frac{1}{8}\cos^2 x \qquad y^8 - 8y + 7 = 0 \qquad y = 1$$
$$\sin^2 x = 1 - \cos^2 x = 1 - 8(1-y)$$

N19. Through a point draw a straight line cutting off a $\triangle$-angle of a given perimeter from a given angle.

N20. Conduct a complete investigation with respect to "a": $4^x + 2 = a \cdot 2^x \sin \pi x$.

N21. Given 4 circles, moreover each of them is tangent to the three others. The centers of three of them lie on one line. The distance from the center of the 4-th to this line - x. Find x, if R - radius of the circle.

N22. On a plane are given a circle with a diameter drawn in it and a point. Drop a perpendicular to the diameter using only a straightedge

N23. Using only a compass divide a segment into n equal parts.

Construct $n \cdot AB$ and perform inversion.

N24. Given three circles with radii a, b, c. Each of them touches the two others, the sides of a $\triangle$-angle, and exscribed circle. Find the radius of the latter.

N25. Can a regular pentagon be obtained as a section of a parallelepiped?

N26. Given a regular $\triangle$-angle and pt. O inside it. $\angle BOC = x$; $\angle AOC = y$. Find the angles of the $\triangle$ with sides AO, BO, CO.

(Rotate around A by an angle of 60° pt. $O \to$ pt. O' $\triangle$ COO' or $\triangle$ BOO' - the required one).

N27. In a quadrilateral O - point inters. diagonals. Perimeters of $\triangle$-angles AOB, BOC, COD, DOA are equal. Prove: $ABCD$ - rhombus.

N28. Quadrilateral $ABCD$ is inscribed in a circle with center O. Line $EF \perp OK$, K lies on EF, E - on AB, F - on CD. Prove that $EK = KF$.

N29. A - radius of the circumscribed, and a - of the inscribed circle in a $\triangle$-angle. The distance between their centers - b. Prove that $A^2 - b^2 = 2Aa$.

N30. Given two intersecting lines. Find the l.o.p. [locus of points] from which the sum of distances to the two given lines is constant.

(rectangle)

N31. The sides of a spatial 4-gon are tangent to a sphere. Prove that the points of tangency lie in one plane.

N32. The faces of a tetrahedron have equal area. Prove, that they are equal.

N33. A 4-gon with sides a, b, c, d is inscribed and circumscribed about circles. Prove, that its area is equal to the root of $abcd$.

For inscr. quad. $S = \sqrt{(p-a)(p-b)(p-c)(p-d)}$

N34. p_n - sequence of the first n prime numbers. Prove, that $\prod_{i=1}^{n} p_i < 4^{p_n}$

N35. There are four villages lying at the vertices of a square. Lay out straight roads so that from each village one could travel to any other and the length of roads is minimal.

N36. The plane is colored in three colors. Prove that $\exists$ two points of one color at dist. 1.

N37. Given a sequence $\{a_n\}$. The limit of the difference of two consecutive terms is equal to 0. Does the limit of the sequence exist

Answer: (Not necessarily. e.g. $a_n = \sum_{k=1}^{n} \frac{1}{k}$

N38. Prove, that sine of 10° is irrational.

(Sol.n:

$$\sin 30^\circ = 3\sin 10^\circ - 4\sin^3 10^\circ \implies 8\sin^3 10^\circ - 6\sin 10^\circ + 1 = 0.$$
$$2\sin 10^\circ = x \implies x^3 - 3x + 1 = 0.$$

All rat. roots - integers and divisors of the const. term 1 but ± 1 do not satisfy $\implies$ roots are irrational).

N39. Given segments a and b. Construct segment x such, that $\sqrt[4]{x} = \sqrt[4]{a} + \sqrt[4]{b}$.

(Sol: $x = a + b + 4\left(\sqrt[4]{a^3b} + \sqrt[4]{ab^3}\right) + 6\sqrt{ab}$).

N40. Solve the eq-tion: $x^{(x^n)} = n$

(Sol:

$$\begin{array}{ll} \text{at } n \geq 1 & x = \sqrt[n]{n} \\ \text{for } x > \sqrt[n]{n} & x^{(x^n)} > n \\ \text{for } x < \sqrt[n]{n} & x^{(x^n)} < n \end{array}$$

)?

N41. Find $\sin 1^\circ$.

(Find $\sin 20^\circ$ and $\sin 18^\circ$).

N42. Is it possible to arrange on a plane 6 points so that the distance between any pair of points would be an integer?

N43. A convex poly-dron has all faces as $\triangle$-angles. Prove, that it has an edge to which 4 acute angles are adjacent.

N44. Sol. ineq-ty: $2^{\sin x} + 2^{\cos x} \geq 2^{1-\frac{1}{\sqrt{2}}}$

Sol: $2^{\sin x} + 2^{\cos x} \geq 2\sqrt{2^{\sin x + \cos x}} = 2 \cdot 2^{\frac{\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)}{2}} = 2^{1+\frac{\sin\left(x+\frac{\pi}{4}\right)}{\sqrt{2}}} \geq 2^{1-\frac{1}{\sqrt{2}}}$

N45. Is it possible to arrange on squared paper a regular $\triangle$-angle with vertices in the nodes.

N46. Two 4-gons are similar, if their corresponding angles and the angle between the diagonals are equal. Prove.

N47. $y = \sqrt[4]{x} + \sqrt[4]{1-x}$ Find the maximum.

N48. Construct a 4-gon given its sides and midline.

N49. Among all 4-gons find the one with the largest area. (sides are given).

N50. Given two parallel segments. Using only a straightedge, divide one of them into six equal parts.

N51. Can an irrational power of an irrational number be rational.

$(\sqrt{2}^{\log_{\sqrt{2}} 3} = 3;\ \log_{\sqrt{2}} 3 = 2\log_2 3)$

N52. Which is larger: $\log_2 3$ or $\log_3 5$

$(\log_3 5 < \frac{3}{2} < \log_2 3)$

N53. $a + b + c = 1$. Prove: $a^2 + b^2 + c^2 \geq \frac{1}{3}$

N54. Find $\operatorname{tg}\frac{\pi}{7} \cdot \operatorname{tg}\frac{2\pi}{7} \cdot \operatorname{tg}\frac{3\pi}{7}$

N55. Is it possible to represent the function $f(x, y) = (xy)^{200} + 1$ in the form of a product of two polynomials (one of x, the other of y)

Let $f(x, y) = P(x) \cdot Q(y)$ Put $x = 0$, then $\forall y\, Q(y) \cdot P(0) = 1 \implies Q(y) = \frac{1}{P(0)}$

N56. $|x| < 0.05$. Find with an accuracy up to 0.01 the value of the expression: $\sqrt[3]{2 - \sqrt{1-x}}$

N57. $\sin\frac{8}{7}\ ?\ \frac{8}{27}\pi$

N58. $3^{\sqrt{2}}\ ?\ 2^{\sqrt{5}}$

(Sol. guessing $\frac{p}{q}$ $\quad \log_2 3\ ?\ \frac{p}{q}\ ?\ \sqrt{\frac{5}{2}}$

N59. C - midpoint of chord AB. Through C are drawn 2 chords, their ends are connected. Prove, that the resulting chords cut off equal segments on AB.

Sol: Let $AC = CB = a$.

$$pq = a^2 - x^2 \qquad p = \frac{x \sin\beta}{\sin\varphi}$$

$$mn = a^2 - y^2 \qquad q = \frac{x \sin\alpha}{\sin(\varphi + \alpha + \beta)}$$

$$pq = \frac{x^2 \sin\alpha \sin\beta}{\sin(\varphi + \alpha + \beta)\sin\varphi} = x^2 A$$

$$mn = y^2 A \implies x^2 = \frac{a^2}{A+1} = y^2$$

N60. $x^3 + 1 = 2\sqrt[3]{2x - 1}$

(Sol: $y = \dfrac{x^3+1}{2}$, $y(x)$ and $x(y)$ - mutually inverse

N61. $a^b + b^a = n$ $\exists$ any a and b - irrat.

Prove $x^x = 3$ and x - irrat.

$$\left(\frac{p}{q}\right)^{\frac{p}{q}} = 3 \qquad \left(\frac{p}{q}\right)^{p} = 3^q \qquad \frac{p^p}{x^p} = \frac{q^p \cdot 2^q}{2^q}$$