Coffin problems

Solutions

Evaluations

Problem 1.1

Let $a_1, \dots, a_n \in \mathbb{C}$ be an arithmetic progression, with $\cos(a_i) \neq 0$ for all $1 \leq i \leq n$. Evaluate the sum:

$$\sum_{i=1}^{n-1} \frac{1}{\cos(a_i)\cos(a_{i+1})}$$

Solution

Let r be the common difference of the arithmetic progression. If r is a multiple of π, then, trivially, letting $k \in \mathbb{Z}$ such that $r = k\pi$:

$$\sum_{i=1}^{n-1} \frac{1}{\cos(a_i)\cos(a_{i+1})} = \sum_{i=1}^{n-1} \frac{1}{(-1)^k \cos^2(a_1)} = (-1)^k \frac{n-1}{\cos^2(a_1)}$$

If r is not a multiple of π, then:

$$\sum_{i=1}^{n-1} \frac{1}{\cos(a_i)\cos(a_{i+1})} = \frac{\sin(a_n - a_1)}{\sin(r)\cos(a_1)\cos(a_n)} = \frac{\tan(a_n) - \tan(a_1)}{\sin(r)}$$

In fact:

$$
\begin{aligned}
\sin(r)\sum_{i=1}^{n-1}\frac{1}{\cos(a_i)\cos(a_{i+1})} &= \sum_{i=1}^{n-1}\frac{\sin(r)}{\cos(a_i)\cos(a_{i+1})} = \\
&= \sum_{i=1}^{n-1}\frac{\sin(a_{i+1}-a_i)}{\cos(a_i)\cos(a_{i+1})} = \\
&= \sum_{i=1}^{n-1}\frac{\sin(a_{i+1})\cos(a_i)-\sin(a_i)\cos(a_{i+1})}{\cos(a_i)\cos(a_{i+1})} = \\
&= \sum_{i=1}^{n-1}\tan(a_{i+1})-\tan(a_i) = \\
&= \tan(a_n)-\tan(a_1)
\end{aligned}
$$

Problem 1.2

Find $\sin(1°)$.

Solution 1

Observe that $1° = 10° - 9°$, and $10° = \frac{\pi}{18}$. Hence:

$$
\begin{aligned}
\sin(1°) &= \sin(10°-9°) = \sin(10°)\cos(9°) - \cos(9°)\sin(9°) = \\
&= \frac{1}{2i}\cos(9°)(e^{i\frac{\pi}{18}} - e^{-i\frac{\pi}{18}}) - \frac{1}{2}\sin(9°)(e^{i\frac{\pi}{18}} + e^{-i\frac{\pi}{18}}) = \\
&= \frac{1}{2}e^{i\frac{\pi}{18}}(-\sin(9°) - i\cos(9°)) + \frac{1}{2}e^{-i\frac{\pi}{18}}(-\sin(9°) + i\cos(9°))
\end{aligned}
$$

The term $e^{i\frac{\pi}{18}}$ is the principal cube root of $e^{i\frac{\pi}{6}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$, and the term $e^{-i\frac{\pi}{18}}$ is the principal cube root of $e^{-i\frac{\pi}{6}} = \frac{\sqrt{3}}{2} - \frac{1}{2}i$, so they can be written as:

$$
e^{i\frac{\pi}{18}} = \sqrt[3]{\frac{\sqrt{3}+i}{2}}, \qquad e^{-i\frac{\pi}{18}} = \sqrt[3]{\frac{\sqrt{3}-i}{2}}
$$

The values of $\sin(9°)$ and $\cos(9°)$ are the following (they can be calculated using the angle difference formulas from $9° = 45° - 36°$):

$$
\begin{aligned}
\sin(9°) &= \frac{1}{8}\sqrt{2}\left(\sqrt{5}+1\right) - \frac{1}{4}\sqrt{5-\sqrt{5}} \\
\cos(9°) &= \frac{1}{8}\sqrt{2}\left(\sqrt{5}+1\right) + \frac{1}{4}\sqrt{5-\sqrt{5}}
\end{aligned}
$$

Altogether:

$$\sin(1^\circ) = \frac{1}{8}\sqrt[3]{\frac{\sqrt{3}+i}{2}} \cdot \left(-\frac{1}{2}\sqrt{2}\left(\sqrt{5}+1\right) + \sqrt{5-\sqrt{5}} \;-\; i\left(\frac{1}{2}\sqrt{2}\left(\sqrt{5}+1\right) + \sqrt{5-\sqrt{5}}\right)\right) \;+$$
$$+\; \frac{1}{8}\sqrt[3]{\frac{\sqrt{3}-i}{2}} \cdot \left(-\frac{1}{2}\sqrt{2}\left(\sqrt{5}+1\right) + \sqrt{5-\sqrt{5}} \;+\; i\left(\frac{1}{2}\sqrt{2}\left(\sqrt{5}+1\right) + \sqrt{5-\sqrt{5}}\right)\right)$$

Solution 2

The complex number $\cos(1^\circ) + i\sin(1^\circ)$ is:

$$\cos(1^\circ) + i\sin(1^\circ) = e^{i\left(\frac{\pi}{18}-\frac{\pi}{20}\right)} = e^{i\frac{\pi}{18}} \cdot \left(\cos(9^\circ) - i\sin(9^\circ)\right) =$$
$$= \frac{1}{4}\sqrt[3]{\frac{\sqrt{3}+i}{2}} \cdot \left(\frac{1}{2}\sqrt{2}\left(\sqrt{5}+1\right) + \sqrt{5-\sqrt{5}} - i\left(\frac{1}{2}\sqrt{2}\left(\sqrt{5}+1\right) - \sqrt{5-\sqrt{5}}\right)\right)$$

Then, $\sin(1^\circ)$ is its imaginary part.

Problem 1.3

Evaluate $\tan\left(\frac{1}{7}\pi\right) \cdot \tan\left(\frac{3}{7}\pi\right) \cdot \tan\left(\frac{5}{7}\pi\right)$.

Solution

To do

Problem 1.4

Determine the number of digits of 125^{100} in base 10.

Solution 1

The number of digits is 210. To prove this, it needs to be shown that $10^{209} \leq 125^{100} < 10^{210}$.

For the first inequality, first observe that $5^5 > 3{\cdot}2^{10}$ (since $5^5 = 3125$, and $3{\cdot}2^{10} = 3{\cdot}1024 = 3072$). Squaring yields: $5^{10} > 9 \cdot 2^{20} > 8 \cdot 2^{20}$. Raising to the power of 10 gives: $5^{100} > 8^{10} \cdot 2^{200} = 1024^3 \cdot 2^{200} > 10^9 \cdot 2^{200}$. Multiplying by 5^{200} yields: $125^{100} = 5^{300} = 5^{200} \cdot 5^{100} > 5^{200} \cdot 10^9 \cdot 2^{200} = 10^{209}$.

The second inequality is equivalent to $125^{10} < 10^{21}$, which follows by: $125^3 \cdot 125^7 < 128^3 \cdot 125^7 = 2^{21} \cdot 5^{21}$.

Solution 2

The number of digits is 210. To prove this, it needs to be shown that $10^{209} \le 125^{100} < 10^{210}$.

For the first inequality, first observe that $5^5 > 3{\cdot}2^{10}$ (since $5^5 = 3125$, and $3{\cdot}2^{10} = 3{\cdot}1024 = 3072$). Squaring yields: $5^{10} > 9 \cdot 2^{20} > 8 \cdot 2^{20} = 2^{23}$. Multiplying by 5^{23} gives: $125^{11} = 5^{33} = 5^{10} \cdot 5^{23} > 2^{23} \cdot 5^{23} = 10^{23}$. Raising to the power of 9 gives $125^{99} > 10^{9 \cdot 23} = 10^{207}$. Finally, $125^{100} = 125 \cdot 125^{99} > 100 \cdot 125^{99} > 100 \cdot 10^{207} = 10^{209}$.

The second inequality is equivalent to $5^{10} < 10^7 = 2^7 \cdot 5^7$ (by raising to the power of 30), which is equivalent to $5^3 < 2^7$, which is true ($125 < 128$).

Solution 3

The number of digits of a positive integer n in base b is $\lfloor \log_b(n) + 1 \rfloor$. Thus, the required number of digits is $\lfloor 100 \log_{10}(125) + 1 \rfloor$, (and the problem is essentially that of evaluating $\log_{10}(125)$ to two decimal places).

First notice that $\log_{10}(2) > \frac{3}{10}$, since $2^{10} = 1024 > 1000 = 10^3$. Then, observe that $3 = \log_{10}(1000) = \log_{10}(2^3 \cdot 5^3) = \log_{10}(125) + 3\log_{10}(2)$. From this, the following upper bound is deduced: $100\log_{10}(125) = 300 - 300\log_{10}(2) < 300 - 300 \cdot \frac{3}{10} = 210$.

Now observe that $\log_{10}(125) = \log_{10}\left(100 \cdot \frac{5}{4}\right) = 2 + \log_{10}\left(\frac{5}{4}\right)$. The following steps show that $100\log_{10}\left(\frac{5}{4}\right) > 9$:

$$\left(\frac{5}{4}\right)^4 = \left(1 + \frac{1}{4}\right)^4 = \left(1 + \frac{1}{2} + \frac{1}{16}\right)^2 > 1 + 1 + \frac{1}{4} + \frac{1}{8} = 2 + \frac{3}{8}$$
$$\left(\frac{5}{4}\right)^8 > \left(2 + \frac{3}{8}\right)^2 > 4 + \frac{3}{2} = 5 + \frac{1}{2}$$
$$\left(\frac{5}{4}\right)^{16} > \left(5 + \frac{1}{2}\right)^2 > 25 + 5 = 30$$
$$\left(\frac{5}{4}\right)^{33} > \frac{5}{4} \cdot 30^2 > 1000$$
$$\left(\frac{5}{4}\right)^{100} > \left(\frac{5}{4}\right)^{99} > 1000^3 = 10^9$$

In conclusion: $100\log_{10}(125) = 200 + 100\log_{10}\left(\frac{5}{4}\right) > 200 + 9 = 209$.
Therefore, $209 < \log_{10}(125^{100}) < 210$. It follows that $\lfloor \log_{10}(125^{100}) + 1 \rfloor = 210$.

Comparisons

Problem 2.1

Determine which among $\log_2(3)$ and $\log_3(5)$ is largest.

Solution

Observe that:

$$
\begin{aligned}
2\log_2(3) &= \log_2(9) > \log_2(8) = 3 \\
2\log_3(5) &= \log_3(25) < \log_3(27) = 3
\end{aligned}
$$

Thus $\log_3(5) < \log_2(3)$.

Problem 2.2

Determine which among $\prod_{n=2}^{40} \log_3(2n)$ and $2\prod_{n=2}^{40} \log_3(2n-1)$ is largest.

Solution

To do

Problem 2.3

Determine which among $\frac{8}{27}\pi$ and $\sin\left(\frac{8}{7}\right)$ is largest.

Solution

To do

Problem 2.4

Determine which among $\sqrt[3]{413}$ and $6 + \sqrt[3]{3}$ is largest.

Solution 1

Claim: $\sqrt[3]{413} > 7 + \frac{4}{9}$. This can be shown as follows:

$$
\begin{aligned}
\left(7 + \frac{4}{9}\right)^3 &= 7^3 + 3 \cdot 7^2 \cdot \frac{4}{9} + 3 \cdot 7 \cdot \frac{4^2}{9^2} + \frac{4^3}{9^3} = 343 + \frac{196}{3} + \frac{112}{27} + \frac{64/27}{27} = \\
&= 343 + 65 + \frac{1}{3} + 4 + \frac{4}{27} + \frac{2 + 10/27}{27} < 412 + \frac{1}{3} + \frac{9}{27} = \\
&= 412 + \frac{2}{3} < 413
\end{aligned}
$$

Claim: $6+\sqrt[3]{3}<7+\dfrac{4}{9}$; that is: $\sqrt[3]{3}<1+\dfrac{4}{9}$. This can be shown as follows:

$$\begin{aligned}\left(1+\frac{4}{9}\right)^3 &= 1+3\cdot\frac{4}{9}+3\cdot\frac{4^2}{9^2}+\frac{4^3}{9^3} = 1+\frac{4}{3}+\frac{16}{27}+\frac{64}{729} = 2+\frac{1}{3}+\frac{9+7}{27}+\frac{64/27}{27} = \\ &= 2+\frac{1}{3}+\frac{1}{3}+\frac{7}{27}+\frac{2+10/27}{27} = 2+\frac{2}{3}+\frac{9}{27}+\frac{10}{729}>3\end{aligned}$$

So, in conclusion:

$$6+\sqrt[3]{3}<7+\frac{4}{9}<\sqrt[3]{413}$$

Solution 2

Let $\alpha=\sqrt[3]{413}-\sqrt[3]{3}$. Then: $\alpha^3=413-3\cdot\sqrt[3]{3}\cdot\sqrt[3]{413}\cdot\left(\sqrt[3]{413}-\sqrt[3]{3}\right)-3=410-3\sqrt[3]{1239}\,\alpha$.

Therefore, α is a root of the polynomial function $p(x)=x^3+3\sqrt[3]{1239}x-410$. This function is strictly increasing, as a function $\mathbb{R}\to\mathbb{R}$, hence α is its only real root, and, moreover, for every $x\in\mathbb{R}$, $x<\alpha$ if and only $p(x)<0$, and $x>\alpha$ if and only $p(x)>0$.

Putting $x=6$, it needs to be determined whether $216+18\sqrt[3]{1239}-410$ is positive or negative, which simplifies to determining whether 97 is, respectively, less or greater than $9\sqrt[3]{1239}$. Now:

$$\begin{aligned}\left(\frac{97}{9}\right)^3 &= \left(10+\frac{7}{9}\right)^3 > 1000+3\cdot100\cdot\frac{7}{9}+3\cdot10\cdot\frac{49}{81} = \\ &= 1000+\frac{700}{3}+\frac{490}{27}>1000+233+10> \\ &> 1239\end{aligned}$$

It follows that $97>9\sqrt[3]{1239}$, so $p(6)<0$, which means that $6<\alpha=\sqrt[3]{413}-\sqrt[3]{3}$. So, in conclusion:

$$6+\sqrt[3]{3}<\sqrt[3]{413}$$

Solution 3

Let the polynomial $p(x,y,z)=(x-y)^2+(y-z)^2+(z-x)^2$, and recall the polynomial identity:

$$x^3+y^3+z^3-3xyz=(x+y+z)\cdot\frac{1}{2}p(x,y,z)$$

Note that the polynomial p evaluated at real values is always non-negative, since it is a sum of squares. Substituting $x=\sqrt[3]{413}$, $y=-6$, $z=-\sqrt[3]{3}$ in the above identity gives:

$$\begin{aligned}\frac{1}{2}\left(\sqrt[3]{413}-\left(6+\sqrt[3]{3}\right)\right)\cdot p\left(\sqrt[3]{413},-6,-\sqrt[3]{3}\right) &= 413-216-3-3\cdot6\cdot\sqrt[3]{3\cdot413} = \\ &= 194-18\sqrt[3]{1239} = \\ &= 2\cdot\left(97-9\sqrt[3]{1239}\right)\end{aligned}$$

Hence, if $97 > 9\sqrt[3]{1239}$, then $\sqrt[3]{413}$ is greater than $6+\sqrt[3]{3}$, and if $97 < 9\sqrt[3]{1239}$, then $\sqrt[3]{413} < 6+\sqrt[3]{3}$. Now:

$$\begin{aligned}\left(\frac{97}{9}\right)^3 &= \left(10+\frac{7}{9}\right)^3 > 1000 + 3\cdot 100\cdot\frac{7}{9} + 3\cdot 10\cdot\frac{49}{81} = \\ &= 1000 + \frac{700}{3} + \frac{490}{27} > 1000 + 233 + 10 > \\ &> 1239\end{aligned}$$

It follows that $97 > 9\sqrt[3]{1239}$. So, in conclusion:

$$6+\sqrt[3]{3} < \sqrt[3]{413}$$

Addendum The values of the two expressions are approximately:

$$\begin{aligned}\sqrt[3]{413} &\approx 7.447034238 \\ 6+\sqrt[3]{3} &\approx 7.442249570\end{aligned}$$

Their difference is approximately 0.004784668.

Problem 2.5

Prove that:

$$\sqrt[3]{3+\sqrt[3]{3}} + \sqrt[3]{3-\sqrt[3]{3}} < 2\sqrt[3]{3}$$

Solution 1

Let $r \in (0,1)$, and consider the function $f : \mathbb{R}^+ \to \mathbb{R}^+ : x \mapsto x^r$. Let $a \in \mathbb{R}^+$, and consider the function $g : \mathbb{R}^+ \to \mathbb{R}^+ : x \mapsto f(x+a) - f(x)$.

Claim: g is decreasing.
This can be proven by showing that $g' < 0$. Now: $g'(x) = f'(x+a) - f'(x)$, so it is sufficient for f' to be decreasing, which is the case if $f'' < 0$. This is the case: $f''(x) = r(r-1)x^{r-2} < 0$, because $r > 0$, $r-1 < 0$, $x > 0$.

Since g is decreasing, then, in particular, $g(x) < g(x-a)$, that is: $f(x+a) - f(x) < f(x) - f(x-a)$, which can be rewritten as $f(x+a) + f(x-a) < 2f(x)$.

With $r = \dfrac{1}{3}$, $a = \sqrt[3]{3}$, and $x = 3$, the claimed inequality follows.

Solution 2

Let $r \in (0,1)$, and consider the function $f : \mathbb{R}^+ \to \mathbb{R}^+ : x \mapsto x^r$.

f is strictly concave (that is, $-f$ is strictly convex), since for all $x \in \mathbb{R}^+$ $f''(x) = r(r-1)x^{r-2} < 0$, because $r > 0$, $r - 1 < 0$, and $x > 0$.
Hence, for any $x, y \in \mathbb{R}^+$ with $x \neq y$, and for any $t \in (0,1)$: $f(tx + (1-t)y) > tf(x) + (1-t)f(y)$.
Putting $r = \frac{1}{3}$, $x = 3 + \sqrt[3]{3}$, $y = 3 - \sqrt[3]{3}$, $t = \frac{1}{2}$, the claimed inequality is obtained.

Problem 2.6

Prove that:

$$\sqrt{3 + 32\sin^2(15°)} + \cos(22°) + \cos(70°) + \cos(88°) + 2\sqrt{2}\sin(15°) > \frac{3}{2}\left(\cos(11°) + \cos(35°) + \cos(44°)\right)^2$$

Solution

To do

Equations

Problem 3.1

For $a \in \mathbb{R}$, find all real numbers $x \geq -a$ that satisfy:

$$\sqrt{a + \sqrt{a + x}} = x$$

Solution

To do

Problem 3.2

Find all $x \in \mathbb{R}$ that satisfy:

$$2\sqrt[3]{2x - 1} = x^3 + 1$$

Solution

To do

Problem 3.3

Let $a \in \mathbb{R}^+$. Find all $x \in \mathbb{R}^+$ that satisfy:

$$x^{x^a} = a$$

Solution

To do

Problem 3.4

Find all $x \in \mathbb{R}$ (or $\mathbb{C}$?) that satisfy:

$$x^4 - 14x^3 + 66x^2 - 115x + 66 + \frac{1}{4} = 0$$

Solution

To do

Problem 3.5

Find all $x, y \in \mathbb{R}$ that satisfy:

$$\begin{cases} y \cdot (x+y)^2 = 9 \\ y \cdot (x^3 - y^3) = 7 \end{cases}$$

Solution

To do

Problem 3.6

For each $n \in \mathbb{N}$, determine the set M_n of pairs $(a, b) \in \mathbb{R}^2$ such that the equation $x^2 - a = |x - b|$ has exactly n solutions in $\mathbb{R}$.

Describe the plot of each set M_n in $\mathbb{R}^2$.

Solution

To do

Problem 3.7

Investigate the following equation, for $x \in \mathbb{R}$, parametrised by $a \in \mathbb{R}$:

$$4^x + 2 = 2^x a \sin(\pi x)$$

Solution

To do

Problem 3.8

Find all $x \in \mathbb{R}$ that satisfy:

$$\sin^7(x) + \frac{1}{\sin^3(x)} = \cos^7(x) + \frac{1}{\cos^3(x)}$$

Solution

To do

Problem 3.9

Find all $x \in \mathbb{R}$ that satisfy:

$$\sin^{\frac{11}{7}}(x) + \cos^{\frac{19}{11}}(x) = \sqrt{\frac{19}{7}}$$

Solution

To do

Problem 3.10

Find all $x \in \mathbb{R}$ that satisfy:

$$\left(1 - \frac{1}{8}\cos^2(x)\right)^8 = \sin^2(x)$$

Solution

Make the substitution $y = 1 - \frac{1}{8}\cos^2(x)$. The range of possibe values of y is $\left[1 - \frac{1}{8}, 1\right] = \left[\frac{7}{8}, 1\right]$. Using that $\sin^2(x) = 1 - \cos^2(x)$ and $\cos^2(x) = 8 - 8y$, the substitution yields the equation

$$y^8 - 8y + 7 = 0$$

Clearly, 1 is a solution, and it yields the factorisation

$$y^8 - 8y + 7 = (y-1)\left(-7 + \sum_{k=1}^{7} y^k\right)$$

The term $-7 + \sum_{k=1}^{7} y^k$ is a strictly increasing function of y for $y > 0$. Thus, in particular, it admits at most one root in the range $\left[\frac{7}{8}, 1\right]$. The root is 1, trivially. Hence, the equation $y^8 - 8y + 7 = 0$ admits only the solution $y = 1$ (with multiplicity 2) in the range $\left[\frac{7}{8}, 1\right]$.

The solution $y = 1$ corresponds to $1 - \frac{1}{8}\cos^2(x) = 1$, that is $\cos(x) = 0$. It is immediate to see also that every $x \in \mathbb{R}$ such that $\cos(x) = 0$ is a solution to the original equation.

In conclusion, the solutions are $x = \frac{\pi}{2} + k\pi$ for $k \in \mathbb{Z}$.

Problem 3.11

Find all $x \in (0, \pi)$ that satisfy:

$$\cot(x) = \sin\left(x + \frac{\pi}{4}\right)$$

Solution

To do

Problem 3.12

Find all $x \in \mathbb{R}$ that satisfy:

$$\sin^3(x)\cos\left(\frac{x}{2}\right) + \frac{1}{2}\sin(x)\sin\left(\frac{x}{2}\right)\left(1 + 2\cos\left(\frac{x}{2}\right)\right) - 6\sin^2\left(\frac{x}{2}\right) - 1 = 0$$

Solution

To do

Problem 3.13

Find all $x \in \mathbb{R}^+$ that satisfy:

$$\frac{1}{16^x} = \log_{\frac{1}{16}}(x)$$

Solution

To do

Inequations

Problem 4.1

Find all $x \in [-1, 1]$ that satisfy:

$$x \cdot \left(8\sqrt{1-x} + \sqrt{1+x}\right) \leq 11\sqrt{1+x} - 16\sqrt{1-x}$$

Solution

Notice that $x = -1$ does not satisfy the inequality; thus the range of values of x can be restricted to $(-1, 1]$. The inequality is equivalent to the following, obtained by dividing by $\sqrt{1+x}$, which is always strictly positive for $x \in (-1, 1]$:

$$x \cdot \left(8\frac{\sqrt{1-x}}{\sqrt{1+x}} + 1\right) \leq 11 - 16\frac{\sqrt{1-x}}{\sqrt{1+x}}$$

Let $y = \frac{\sqrt{1-x}}{\sqrt{1+x}}$. Hence $x = \frac{1-y^2}{1+y^2}$. The inequality becomes:

$$\frac{1-y^2}{1+y^2} \cdot (8y+1) \leq 11 - 16y$$

and it is equivalent to the following, obtained by multiplying by $1 + y^2$ (which is always strictly positive):

$$(1-y^2)(8y+1) \leq (1+y^2)(11-16y)$$

which, expanded, is:

$$8y^3 - 12y^2 + 24y - 10 \leq 0$$

The polynomial $8y^3 - 12y^2 + 24y - 10$ has a root at $y = \frac{1}{2}$, which yields the factorisation $8y^3 - 12y^2 + 24y - 10 = 2(2y-1)(2y^2 - 2y + 5)$. Thus, the inequality can be written as:

$$2(2y-1)(2y^2 - 2y + 5) \leq 0$$

The quantity $2y^2 - 2y + 5$ is always strictly positive, so the inequality is equivalent to $2y - 1 \leq 0$, that is: $2\frac{\sqrt{1-x}}{\sqrt{1+x}} \leq 1$. Since $\frac{\sqrt{1-x}}{\sqrt{1+x}}$ is non-negative, the inequality is equivalent to $4\frac{1-x}{1+x} \leq 1$, obtained by squaring. Since $x > -1$, the inequality is equivalent to $4(1-x) \leq 1 + x$, obtained by multiplying by $1 + x$, which is positive. This simplifies to $5x \geq 3$, that is: $x \geq \frac{3}{5}$.

In conclusion, the solution set is $\left[\frac{3}{5}, 1\right]$.

Problem 4.2

Find all $a \in \mathbb{R}$ such that for any $x \in \mathbb{R}^+$ the following holds:

$$ax^2 + 2x > 3a - 1$$

Solution

To do

Problem 4.3

Find all $x \in \mathbb{R}$ that satisfy:

$$2^{\sin(x)} + 2^{\cos(x)} \geq 2^{1-\frac{1}{\sqrt{2}}}$$

Solution

To do

Problem 4.4

Show that for all $x \in \left(0, \frac{\pi}{2}\right)$ the following holds:

$$\frac{1}{\sin^2(x)} < \frac{1}{x^2} + 1 - \frac{4}{\pi^2}$$

Solution

To do

Problem 4.5

Show that for all $x \in \left(0, \frac{\pi}{2}\right]$ the following holds:

$$\frac{\sin^3(x)}{x^3} > \cos(x)$$

Solution

To do

Problem 4.6

Find all $(x, y) \in (-3, 3) \times \mathbb{R}$ that satisfy:

$$3^y \log_3(9 - x^2) \leq 1 + 3^{2y}$$

Solution

The inequality can be equivalently written as

$$\log_3(9 - x^2) \leq 3^y + 3^{-y}$$

It is satisfied by all $(x, y) \in (-3, 3) \times \mathbb{R}$. In fact, $\log_3(9 - x^2)$ is maximum when $9 - x^2$ is maximum, which is when $x = 0$; this corresponds to a global maximum of $\log_3(9) = 2$.

And $3^y + 3^{-y}$ is minimum for $y = 0$, (since $3^y + 3^{-y} = \left(3^{y/2} - 3^{-y/2}\right)^2 + 2$) which corresponsds to a global minimum of $3^0 + 3^0 = 2$.

Therefore, for every x and y, the inequality $\log_3(9 - x^2) \leq 2 \leq 3^y + 3^{-y}$ holds.

Algebra and Number Theory

Problem 5.1

Prove that the set $\left\{n + m\sqrt{2} \,\middle|\, n, m \in \mathbb{Z}\right\}$ is dense in $\mathbb{R}$.

Solution

To do

Problem 5.2

For each $a \in \mathbb{N}$, let $\mathrm{P}(a)$ be the set of prime divisors of a. Let:

$$S = \left\{(a, b) \in \mathbb{N}^2 \,\middle|\, a \neq b,\ \mathrm{P}(a) = \mathrm{P}(b),\ \mathrm{P}(a+1) = \mathrm{P}(b+1)\right\}$$

Determine whether S is finite or infinite.

Solution

For every $n \in \mathbb{Z}^+$, let

$$a_n = 2^{n+1} - 2 = 2 \cdot (2^n - 1), \qquad b_n = 2^{n+1} a_n = 2^{2n+2} - 2^{n+2}$$

and observe that $b_n + 1 = (2^{n+1} - 1)^2 = (a_n + 1)^2$. Now:

- $a_n \neq b_n$, since clearly $b_n > a_n$;
- $\mathrm{P}(a) = \{2\} \cup \mathrm{P}(2^n - 1)$ and $\mathrm{P}(b) = \{2\} \cup \mathrm{P}(a) = \mathrm{P}(a)$;
- $\mathrm{P}(b+1) = \mathrm{P}\left((a+1)^2\right) = \mathrm{P}(a+1)$.

In conclusion, the infinite family $\left\{(a_n, b_n)\right\}_{n \in \mathbb{Z}^+}$ is a subset of the set S, which is therefore infinite.

Problem 5.3

For each $i \in \mathbb{Z}^+$, let p_i be the i-th prime number. Prove that for every $n \in \mathbb{Z}^+$:

$$\prod_{i=1}^{n} p_i < 4^{p_n}$$

Solution

To do

Problem 5.4

Prove that $\sin(10°)$ is irrational.

Solution

The relation $\sin(3\theta) = -4\sin^3(\theta) + 3\sin(\theta)$ holds for every $\theta \in \mathbb{R}$. Putting $\theta = 30°$ yields the relation $\frac{1}{2} = \sin(30°) = -4\sin^3(10°) + 3\sin(10°)$. Hence, $\sin(10°)$ is a root of the polynomial $8x^3 - 6x + 1$. By the rational root theorem, the only possible rational roots of that polynomial are $1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{8}$. By inspection, none of those is a root of the polynomial. It follows that $\sin(10°)$ is irrational.

Problem 5.5

Do there exist irrational numbers $a, b \in \mathbb{R}^+$ such that a^b is rational?
Do there exist irrational numbers $a, b \in \mathbb{R}^+$ such that a^b is irrational?

Solution 1

The numbers $a = \sqrt{2}$ and $b = 2\log_2(3)$ are irrational, and $a^b = 3$ is rational.

The numbers $a = \sqrt{2}$ and $b = \log_2(3)$ are irrational, and $a^b = \sqrt{3}$ is irrational.

(The numbers $\log_2(3)$ and $2\log_2(3)$ are irrational because no power of 2 is a power of 3, with non-zero integer exponents).

Solution 2

If $\sqrt{2}^{\sqrt{2}}$ is rational, then $a = \sqrt{2}$ and $b = \sqrt{2}$ are irrational numbers such that a^b is rational; otherwise, $\sqrt{2}^{\sqrt{2}}$ is irrational, so $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$ are irrational numbers such that $a^b = 2$ is rational.

For any $a \in \mathbb{R}^+ \setminus \{1\}$, the function $r : \mathbb{R} \setminus \mathbb{Q} \to \mathbb{R} : x \mapsto a^x$ is a bijection on its image, hence $\#(\mathrm{im}(r)) = \#(\mathbb{R} \setminus \mathbb{Q}) = \#(\mathbb{R}) > \#(\mathbb{Q})$; in particular $\mathrm{im}(r) \nsubseteq \mathbb{Q}$. So, for every irrational $a > 0$, there exist a continuum of irrational numbers $b > 0$ such that $a^b \in \mathbb{R} \setminus \mathbb{Q}$.

Problem 5.6

The digit expansion of a number $a \in (0, 1)$ has 0 as first digit, then for every $n \in \mathbb{N}$, the digits $(2^n + 1)$-th to 2^{n+1}-th are the opposite of the digits 1-st to 2^n-th, respectively, where the opposite of the digit 1 is the digit 0, and viceversa. Prove that a is irrational.

Solution

To do

Problem 5.7

Does there exist a (non-degenerate) equilateral triangle in $\mathbb{R}^2$ whose vertices are rational points? (That is, whose vertices are in $\mathbb{Q}^2$).

Solution

Suppose that there is an equilateral triangle with rational vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$. Its area is $\frac{1}{2}|x_1y_2 - x_1y_3 + x_2y_3 - x_2y_1 + x_3y_1 - x_3y_2|$, which is rational. And its side length l is such that $l^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$, so l^2 is rational.
The area of the triangle is also $\frac{1}{4}l^2\sqrt{3}$, which is irrational (because l^2 is rational and $\sqrt{3}$ is irrational). This is a contradiction, thus proving that an equilateral triangle cannot have all vertices on rational points.

Problem 5.8

Does there exist a point inside a square of side length 1, such that its distances to the four vertices of the square are all rational?

Solution

To do

Problem 5.9

Does there exist a set $S \subseteq \mathbb{R}^2$ of six points, no three of which are collinear, and such that for every $p, q \in S$ the distance $\mathrm{d}(p, q)$ is integer?

Solution

To do

Problem 5.10

Determine all pairs of positive integers (a, b) such that $a^b = b^a$.

Solution

To do

Problem 5.11

Do there exist infinitely many pairs $(a, b) \in \mathbb{N}^2$ such that $a^2 + (a+1)^2 = b^2$?

Solution 1

The following is an infinite sequence of solutions (a_n, b_n):

$$\begin{cases} (a_0, b_0) & = (0, 1) \\ (a_{n+1}, b_{n+1}) & = (3a_n + 2b_n + 1,\ 4a_n + 3b_n + 2) \end{cases}$$

This can be seen by induction on n: for $n = 0$ it is trivial; then, assuming that the relation holds for n:

$$\begin{aligned} a_{n+1}^2 + (a_{n+1}+1)^2 &= 2a_{n+1}^2 + 2a_{n+1} + 1 = \\ &= 2(3a_n + 2b_n + 1)^2 + 2(3a_n + 2b_n + 1) + 1 = \\ &= 18a_n^2 + 8b_n^2 + 24a_n b_n + 18a_n + 12b_n + 5 \\ b_{n+1}^2 &= (4a_n + 3b_n + 2)^2 = \\ &= 16a_n^2 + 9b_n^2 + 24a_n b_n + 16a_n + 12b_n + 4 \end{aligned}$$

They are equal, since their difference is $b_n^2 - 2a_n^2 - 2a_n - 1$, which is 0, by the inductive hypothesis.

Solution 2

Let $c = 2a + 1$. The equation can equivalently be written as:

$$c^2 + 1 = 2b^2$$

This is a Pell equation, which admits infinitely many integer solutions. Note that c is necessarily odd, since $c^2 = 2b^2 - 1$ is odd; so, in the corresponding solutions (a, b), a is indeed an integer.

Solution 3

For positive integers $n > m$, the triples $(n^2 - m^2, 2nm, n^2 + m^2)$ are Pythagorean triples, with different pairs (n, m) yielding different triples. The requirement $n^2 - m^2 = 2nm \pm 1$ can be rewritten as $(n+m)^2 = 2n^2 \pm 1$. Notice that this automatically implies $m < n$ (since otherwise $(n+m)^2 \geq 4n^2 > 2n^2 + 1$).

Let S_+ be the set of pairs $(r, s) \in \mathbb{N}^2$ such that $r^2 = 2s^2 + 1$, and S_- be the set of pairs $(r, s) \in \mathbb{N}^2$ such that $r^2 = 2s^2 - 1$. Observe that if $(r, s) \in S_+$, then $(r + 2s, r + s) \in S_-$, and if $(r, s) \in S_-$, then $(r + 2s, r + s) \in S_+$:

- if $r^2 = 2s^2 + 1$, then $(r+2s)^2 = r^2 + 4rs + 4s^2 = 2r^2 + 4rs + 2s^2 - 1 = 2(r+s)^2 - 1$;
- if $r^2 = 2s^2 - 1$, then $(r+2s)^2 = r^2 + 4rs + 4s^2 = 2r^2 + 4rs + 2s^2 + 1 = 2(r+s)^2 + 1$.

Since $(1, 0) \in S_+$, there is an infinite sequence of distinct elements (r, s) in $S = S_+ \cup S_-$. For each of them, the pair $(n, m) = (r - s, s)$ determines a distinct Pythagorean triple of the desired form. Each solution (a, b) to the given equation corresponds to the two Pythagorean triples $(a, a+1, b)$ and $(a+1, a, b)$, so there are still infinitely many solutions (a, b).

Solution 4

Let $b = a + n$. The equation can be equivalently rewritten as follows (by completing the square):

$$(a - n + 1)^2 = 2n(n-1)$$

It follows that for an integer $n > 0$ there is a solution a if and only if $2n(n-1)$ is a square; since n and $n-1$ do not have any common factors, and one of them is even, this is equivalent to

- n and $2(n-1)$ are perfect squares, when n is odd;
- $2n$ and $n-1$ are perfect squares, when n is even.

In the first case, $n-1$ is twice a square, and n is a square, so there are $m, r \in \mathbb{N}$ such that $n = m^2 = 2r^2+1$; let S_+ be the set of pairs $(m, r) \in \mathbb{N}^2$ such that $m^2 = 2r^2 + 1$. In the second case, n is twice a square, and $n-1$ is a square, so there are $m, r \in \mathbb{N}$ such that $n - 1 = m^2 = 2r^2 - 1$; let S_- be the set of pairs $(m, r) \in \mathbb{N}^2$ such that $m^2 = 2r^2 - 1$. Observe that if $(m, r) \in S_+$, then $(m + 2r, m + r) \in S_-$, and if $(m, r) \in S_-$, then $(m + 2r, m + r) \in S_+$:

- if $m^2 = 2r^2 + 1$, then $(m + 2r)^2 = m^2 + 4mr + 4r^2 = 2m^2 + 4mr + 2r^2 - 1 = 2(m + r)^2 - 1$;
- if $m^2 = 2r^2 - 1$, then $(m + 2r)^2 = m^2 + 4mr + 4r^2 = 2m^2 + 4mr + 2r^2 + 1 = 2(m + r)^2 + 1$.

Since $(1, 0) \in S_+$, there is an infinite sequence of distinct elements of $S = S_+ \cup S_-$, each of which corresponds to a distinct n, and thus to a distinct solution (a, b).

Solution 5

The following is a sequence of solutions (a_n, b_n), defined inductively:

$$\begin{cases} (a_{-1}, b_{-1}) & = (-1, 1) \\ (a_0, b_0) & = (0, 1) \\ (a_{n+1}, b_{n+1}) & = (6a_n - a_{n-1} + 2,\ 6b_n - b_{n-1}) \end{cases}$$

To show it, first observe that for each integer $n > 0$: $a_n > a_{n-1}$ and $b_n > b_{n-1}$.
TODO

Then observe that for each n, this relation holds: $(2a_n + 1)(2a_{n-1} + 1) = 2b_n b_{n-1} - 3$. It can be shown by induction on n: for $n = 0$ it is trivial; then, inductively, assuming it holds for n:

$$\begin{aligned} (2a_{n+1} + 1)(2a_n + 1) &= (2(6a_n - a_{n-1} + 2) + 1)(2a_n + 1) = (12a_n - 2a_{n-1} + 5)(2a_n + 1) = \\ &= 6(2a_n + 1)^2 - (2a_{n-1} + 1)(2a_n + 1) = 6(4a_n^2 + 4a_n + 1) - 2b_n b_{n-1} + 3 = \\ &= 12(2a_n^2 + 2a_n + 1) - 6 - 2b_n b_{n-1} + 3 = 12b_n^2 - 2b_n b_{n-1} - 3 = \\ &= 2(6b_n - b_{n-1})b_n - 3 = 2b_{n+1} b_n - 3 \end{aligned}$$

Now, showing that each (a_n, b_n) satisfies $b_n^2 = a_n^2 + (a_n + 1)^2 = 2a_n^2 + 2a_n + 1$ can be done by induction on n. For $n = -1$ and $n = 0$ it is trivial. Then, assuming that the relation holds for n:

$$\begin{aligned}
a_{n+1}^2 + (a_{n+1} + 1)^2 &= (6a_n - a_{n-1} + 2)^2 + (6a_n - a_{n-1} + 3)^2 = \\
&= 72a_n^2 + 60a_n + 2a_{n-1}^2 - 10a_{n-1} - 24a_n a_{n-1} + 13 = \\
&= 72a_n^2 + 72a_n + 2a_{n-1}^2 + 2a_{n-1} - 12a_n - 12a_{n-1} - 24a_n a_{n-1} + 12 + 1 \\
b_{n+1}^2 &= (6b_n - b_{n-1})^2 = 36b_n^2 + b_{n-1}^2 - 12b_n b_{n-1} = \\
&= 36(2a_n^2 + 2a_n + 1) + (2a_{n-1}^2 + 2a_{n-1} + 1) - 12b_n b_{n-1} = \\
&= 72a_n^2 + 72a_n + 2a_{n-1}^2 + 2a_{n-1} - 12b_n b_{n-1} + 36 + 1
\end{aligned}$$

The two are equal, thanks to the previous relation $(2a_n + 1)(2a_{n-1} + 1) = 2b_n b_{n-1} - 3$.

Problem 5.12

Determine all $a, b, c, d \in \mathbb{Q}$ such that

$$\left(a + b\sqrt{2}\right)^2 + \left(c + d\sqrt{2}\right)^2 = 5 + 4\sqrt{2}$$

Solution

To do

Problem 5.13

Rationalise the denominator in the following fraction:

$$\frac{1}{\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}}$$

Solution

Let $\alpha = \sqrt[3]{a}$, $\beta = \sqrt[3]{b}$, $\gamma = \sqrt[3]{c}$. Consider the polynomial identity:

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

Let the polynomial $p(x, y, z) = x^2 + y^2 + z^2 - xy - yz - zx$. Then:

$$\frac{1}{\alpha + \beta + \gamma} = \frac{1}{\alpha + \beta + \gamma} \cdot \frac{p(\alpha, \beta, \gamma)}{p(\alpha, \beta, \gamma)} = \frac{p(\alpha, \beta, \gamma)}{\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma} = \frac{p(\alpha, \beta, \gamma)}{a + b + c - 3\sqrt[3]{abc}}$$

The new denominator contains only one radical. Let $u = a + b + c$ and $v = 3\sqrt[3]{abc}$. Now:

$$\frac{1}{\alpha+\beta+\gamma} = \frac{p(\alpha,\beta,\gamma)}{u-v} \cdot \frac{u^2+uv+v^2}{u^2+uv+v^2} = \frac{(\alpha^2+\beta^2+\gamma^2-\alpha\beta-\beta\gamma-\gamma\alpha)\cdot(u^2+uv+v^2)}{u^3-v^3}$$

Now the denominator is free of radicals, and, in conclusion, the fraction can be written as:

$$\frac{\left(\sqrt[3]{a^2}+\sqrt[3]{b^2}+\sqrt[3]{c^2}-\sqrt[3]{ab}-\sqrt[3]{bc}-\sqrt[3]{ca}\right)\cdot\left((a+b+c)^2+3(a+b+c)\sqrt[3]{abc}+9\sqrt[3]{a^2b^2c^2}\right)}{(a+b+c)^3-27abc}$$

Analysis

Problem 6.1

Let $f : \mathbb{R} \to \mathbb{R}$ be a monotonically increasing function. Let $a, b \in \mathbb{R}$ with $a < b$. Determine which points $c \in (a, b)$ minimize the value:

$$\int_a^c f(x) - f(a)\,\mathrm{d}x + \int_c^b f(b) - f(x)\,\mathrm{d}x$$

Solution

To do

Problem 6.2

Let $f : [0,1] \to \mathbb{R}^+$ be a continuous function. Let $a, b \in \mathbb{R}^+$ such that $a \le f \le b$. Prove that:

$$ab\int_0^1 \frac{1}{f(x)}\,\mathrm{d}x \le a + b - \int_0^1 f(x)\,\mathrm{d}x$$

Solution

To do

Problem 6.3

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for any $x, y \in \mathbb{R}$ the following holds:

$$f(x) - f(y) \le (x - y)^2$$

Solution 1

Let f be a function satisfying the stated condition. Let $x, y \in \mathbb{R}$. Then $f(x) - f(y) \le (x - y)^2$, and also $f(y) - f(x) \le (y - x)^2 = (x - y)^2$. Hence, for any $x, y \in \mathbb{R}$, the relation $|f(x) - f(y)| \le (x - y)^2$ holds. It follows that f is differentiable and its derivative is 0 everywhere, since for any $x \in \mathbb{R}$:

$$\lim_{y \to x} \left| \frac{f(x) - f(y)}{x - y} \right| \le \lim_{y \to x} |x - y| = 0$$

Therefore, every function f that satisfies the given condition is constant. Viceversa, every constant function clearly satisfies the condition.

Solution 2

Let f be a function satisfying the stated condition. Observe that for any $x, y \in \mathbb{R}$ it holds that $|f(x) - f(y)| \le (x - y)^2$, since $f(x) - f(y) \le (x - y)^2$ and $f(y) - f(x) \le (y - x)^2 = (x - y)^2$.

Let $x, y \in \mathbb{R}$. Let $\varepsilon > 0$. Let $n \in \mathbb{N}$ such that $n > \frac{(y-x)^2}{\varepsilon}$. Let $x_0, \dots, x_n$ be a uniform partition of the interval between x and y: $x_0 = x$, and for each $k \in \{1, \dots, n\}$ $x_k = x_{k-1} + \frac{y-x}{n}$. In particular, $x_n = y$. Then:

$$|f(y) - f(x)| \le \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \le \sum_{k=1}^{n} (x_k - x_{k-1})^2 = n \cdot \left(\frac{y - x}{n} \right)^2 = \frac{(y - x)^2}{n} < \varepsilon$$

So, for any $x, y \in \mathbb{R}$ and for any $\varepsilon > 0$ it holds that $|f(y) - f(x)| < \varepsilon$. Hence for any $x, y \in \mathbb{R}$, $f(y) = f(x)$. This means that f is constant. Therefore, every function f that satisfies the given condition is constant. Viceversa, every constant function clearly satisfies the condition.

Problem 6.4

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for any $x \in \mathbb{R}$ the following holds:

$$f(f(x)) = x^2 - 2$$

Solution

To do

Problem 6.5

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. If $\lim\limits_{n\to+\infty}(a_{n+1}-a_n)=0$, then does $\lim\limits_{n\to+\infty} a_n$ exist (finite or infinite)?

Solution

To do

Problem 6.6

Prove that:

$$\sum_{n=1}^{1000}\frac{1}{n^3+3n^2+2n}<\frac{1}{4}$$

Solution

For each $N\in\mathbb{N}$, using partial fraction decomposition:

$$\begin{aligned}\sum_{n=1}^{N}\frac{1}{n^3+3n^2+2n} &= \sum_{n=1}^{N}\frac{1}{n(n+1)(n+2)} = \frac{1}{2}\sum_{n=1}^{N}\frac{1}{n}-\frac{2}{n+1}+\frac{1}{n+2} = \\ &= \frac{1}{2}\sum_{n=1}^{N}\frac{1}{n}+\frac{1}{2}\sum_{n=3}^{N+2}\frac{1}{n}-\sum_{n=2}^{N+1}\frac{1}{n} = \\ &= \frac{1}{2}+\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{N+1}+\frac{1}{2}\cdot\frac{1}{N+2}-\frac{1}{2}-\frac{1}{N+1}+\sum_{n=3}^{N}\frac{1}{n}-\frac{1}{n} = \\ &= \frac{1}{4}-\frac{1}{2}\cdot\frac{1}{N+1}+\frac{1}{2}\cdot\frac{1}{N+2}+0 = \\ &= \frac{1}{4}-\frac{1}{2(N+1)(N+2)}<\frac{1}{4}\end{aligned}$$

Problem 6.7

Prove that if $a,b,c,d\in\mathbb{R}$ are such that $a^2+4b^2=4$ and $cd=4$, then $(a-d)^2+(b-c)^2\geq\frac{8}{5}$.

Solution

To do

Plane Geometry

Problem 7.1

Let ABC be a triangle, with $\widehat{ABC} = 80°$. Let O be a point inside ABC such that $\widehat{OAC} = 10°$ and $\widehat{OCA} = 30°$.
Express the angle $\widehat{ABO}$ in terms of $\frac{\overline{OB}}{\overline{AC}}$.

Solution

To do

Problem 7.2

Prove that if the lengths of the angle bisectors of a triangle are all less than or equal to 1, then the area of the triangle is less than or equal to $\frac{\sqrt{3}}{3}$.

Solution

To do

Problem 7.3

Four circles on a plane are mutually tangent to each other. The points of tangency are all distinct. Three of the circles have collinear centers. Determine the distance between the center of fourth circle and the line through the centers of the others, in terms of the radius of the fourth circle.
(There are two cases: one for internal tangency and one for external tangency)

Solution

To do

Problem 7.4

Let ABC be a triangle. Let γ be its circumcircle. Let $\alpha_1, \alpha_2, \alpha_3$ be circles such that α_1 is tangent to $\overline{BC}$, to $\overline{CA}$, and to γ (internally); α_2 is tangent to $\overline{AB}$, to $\overline{CA}$, and to γ (internally); α_3 is tangent to $\overline{AB}$, to $\overline{BC}$, and to γ (internally). Determine the radius of γ, given the radii of $\alpha_1, \alpha_2, \alpha_3$.

Alernative formulation.

Three circles are each tangent to a (distinct) unordered pair of (distinct) sides of a triangle and to the circumcircle of the triangle (internally). Determine the radius of the circumcircle, given the radii of the three circles.

Solution

To do

Problem 7.5

Let ABC be an equilateral triangle, and let O be a point inside it. Show that the lengths $\overline{AO}$, $\overline{BO}$, $\overline{CO}$ can be the side lengths of a triangle, and determine the measures of the internal angles in such triangle, in terms of $\widehat{AOB}$ and $\widehat{BOC}$.

Solution

To do

Problem 7.6

Prove that a quadrilateral $ABCD$ is a rhombus if and only if the triangles AOB, BOC, COD, DOA are isoperimetetric, where O is the intersection of the diagonal lines AC and BD.

Solution

To do

Problem 7.7

Given a triangle, let R be the radius of its circumscribed circle, and r the radius of its inscribed circle. Determine the distance s between the centers of these two circles.

Determine the set of possible values of s among all triangles that have a fixed circumradius R; when are the extremes achieved?

Solution

To do

Problem 7.8

Given two intersecting lines r, s on a plane, and a real number $a \geq 0$, find the locus of points P of the plane such that $\mathrm{d}(P, r) + \mathrm{d}(P, s) = a$.

Solution

It is (the boundary of) a rectangle, whose vertices are the two points on r at distance a from s, and the two points on s at distance a from r, and, in particular, the diagonals of the rectangle lie on the lines r and s.

Those four points are clearly part of the locus, since they are at distance 0 from one line and at distance a from the other.

Now to prove that it is a rectangle:

TODO

Now to prove that the sides are in the locus:

TODO

Now to prove that every point of the locus is on one of the sides of the rectangle: In fact, the interior of the rectangle is the set of points such that the sum of distances is $< a$; and its exterior is the set of points such that the sum of distances is $> a$.

Prove that the points in the interior of the rectangle have $< a$:

TODO

Prove that the points outside have $> a$:

TODO

Problem 7.9

Prove that the area of a quadrilateral with side lengths a, b, c, d which admits both inscribed and circumscribed circles is $\sqrt{abcd}$.

Solution

To do

Problem 7.10

Determine the shortest networks that connect the four vertices of a square to each other.

Solution

To do

Problem 7.11

Let $U = \{A, B, C\}$ be a partition of $\mathbb{R}^2$. Prove that for any $a \in \mathbb{R}^+$ there exists $S \in U$ such that there exist $p, q \in S$ with $\mathrm{d}(p, q) = a$.

Solution

To do

Problem 7.12

Prove that in a cyclic quadrilateral, the perpendiculars to each side passing through the midpoint of the opposite side are concurrent.

Solution

To do

Problem 7.13

Determine the quadrilateral with the largest area, given the lengths of its four sides, in order.

Solution

To do

Problem 7.14

Let $\overline{AB}$ be a chord in a circle, and let M be its midpoint. Let $\overline{CD}$ and $\overline{EF}$ be two other chords in the circle that pass through the point M, with C and F on opposite sides of $\overline{AB}$. Prove that $\overline{CF}$ intersects $\overline{AB}$ at a point P, and $\overline{ED}$ intersects $\overline{AB}$ at a point Q, on opposite sides of M, such that $\overline{MP} \cong \overline{MQ}$.

Solution

To do

Problem 7.15

Given a triangle, determine a line that halves both its area and its perimeter.

Solution

To do

Problem 7.16

Let $ABCD$ be a trapezoid with bases $\overline{AB}$ and $\overline{CD}$. Given a point $P \in \overline{AB}$, determine two points $Q_1, Q_2 \in \overline{CD}$ that, respectively, maximize the area of the quadrilateral intersection of the triangles ABQ_1 and CDP, and minimize the area of the quadrilateral intersection of the triangles ABQ_2 and CDP.

Solution

To do

Problem 7.17

Let a, b, c be the side lengths of a triangle, and let α, β, γ be the measures of their opposite angles, respectively. Prove that:

$$\frac{b+c-2a}{\sin\left(\frac{\alpha}{2}\right)} + \frac{c+a-2b}{\sin\left(\frac{\beta}{2}\right)} + \frac{a+b-2c}{\sin\left(\frac{\gamma}{2}\right)} \geq 0$$

Solution

To do

Problem 7.18

Let α, β, γ be the measures of the internal angles of a triangle. Prove that:

$$\sqrt{\sin(\alpha)} + \sqrt{\sin(\beta)} + \sqrt{\sin(\gamma)} \leq \frac{9}{\sqrt{10}}$$

Solution

To do

Problem 7.19

Let a, b, c be the side lengths of a triangle, and let α, β, γ be the measures of their opposite angles, respectively. Prove that:

$$\frac{\pi}{3} \leq \frac{a \cdot \alpha + b \cdot \beta + c \cdot \gamma}{a + b + c} \leq \frac{\pi}{2}$$

Solution

To do

Problem 7.20

Let P be a point in an equilateral (or regular?) hexagon of side length 1. Prove that the sum of the distances between P and the vertices of the hexagon is less than or equal to $4 + 2\sqrt{3}$.

Solution

To do

Problem 7.21

A circle is given on a plane. Given two points on the plane, outside the circle, construct a circle that passes through those two points and is tangent to the first circle.
(Is the problem asking for a straightedge and compass construction?)

Solution

Consider a coordinate system with the two given points lying on the horizontal axis at coordinates $A : (-a, 0)$ and $B : (a, 0)$, for some $a \neq 0$. Let $C : (b, c)$ be the center of the given circle, and let $s > 0$ be the radius.
The wanted circle passes through A and B, so its center must lie on the perpendicular bisector of the two points, which is the vertical axis; so it has coordinates $P : (0, d)$, for some $d \in \mathbb{R}$. Let r be its radius. The tangency condition between the two circles is equivalent to $\mathrm{d}(P, C) = r + s$ (for external tangency) or $\mathrm{d}(P, C) = r - s$ (when the first circle is internally tangent to the wanted circle) or $\mathrm{d}(P, C) = s - r$ (when the wanted circle is internally tangent to the first circle). All in all, the tangency conditions can be summarised as $\mathrm{d}(P, C)^2 = (r \pm s)^2$. Together with the condition $r = \mathrm{d}(P, A) = \mathrm{d}(P, B)$, this yields the following system of equations, in the unknowns d and r:

$$\begin{cases} r^2 = a^2 + d^2 \\ r^2 + s^2 \pm 2rs = b^2 + (c - d)^2 \end{cases}$$

Substituting $a^2 + d^2$ for r^2 in the second equation, and canceling the term d^2 yields:

$$a^2 + s^2 \pm 2rs = b^2 + c^2 - 2cd$$

So:

$$\mp r = \frac{c}{s} d + \frac{1}{2s}(a^2 - b^2 - c^2 + s^2)$$

Substituting in the first equation yields a quadratic equation in d:

$$(c^2 - s^2)d^2 + c(a^2 - b^2 - c^2 + s^2)d + \frac{1}{4}\left(a^2 - b^2 - c^2 + s^2\right)^2 - a^2 s^2 = 0$$

If $c^2 = s^2$, the equation is actually linear:

$$c(a^2 - b^2)d + \frac{1}{4}\left(a^2 - b^2\right)^2 - a^2 s^2 = 0$$

Note that $a^2 - b^2$ cannot be 0, since it would imply $a = 0$ or $s = 0$. Also, $c \neq 0$, since $c^2 = s^2 \neq 0$. The solution in this case is then:

$$d = \frac{a^2 s^2 - \frac{1}{4}\left(a^2 - b^2\right)^2}{c(a^2 - b^2)} = \frac{a^2 c}{a^2 - b^2} - \frac{a^2 - b^2}{4c}$$

Observe that $c^2 = s^2$ if and only if the line AB is tangent to the first circle. In this case, the line AB can be considered a degenerate second solution circle.

In the case $c^2 \neq s^2$, the solutions are:

$$d = \frac{-c\left(a^2 - b^2 - c^2 + s^2\right) \pm \sqrt{c^2\left(a^2 - b^2 - c^2 + s^2\right)^2 - (c^2 - s^2)\left(a^2 - b^2 - c^2 + s^2\right)^2 + 4(c^2 - s^2)a^2 s^2}}{2(c^2 - s^2)}$$

$$= \frac{-c\left(a^2 - b^2 - c^2 + s^2\right) \pm s\sqrt{\left(a^2 - b^2 - c^2 + s^2\right)^2 + 4a^2c^2 - 4a^2 s^2}}{2(c^2 - s^2)} =$$

$$= \frac{-c\left(a^2 - b^2 - c^2 + s^2\right) \pm s\sqrt{\left(a^2 + b^2 + c^2 - s^2\right)^2 - 4a^2 b^2}}{2(c^2 - s^2)} =$$

$$= \frac{-c\left(a^2 - b^2 - c^2 + s^2\right) \pm s\sqrt{\left((a+b)^2 + c^2 - s^2\right)\left((a-b)^2 + c^2 - s^2\right)}}{2(c^2 - s^2)} =$$

$$= \frac{-c\left(a^2 - b^2 - c^2 + s^2\right) \pm s\sqrt{\left(\mathrm{d}(A,C)^2 - s^2\right)\left(\mathrm{d}(B,C)^2 - s^2\right)}}{2(c^2 - s^2)}$$

Observe that $\left((a+b)^2 + c^2 - s^2\right)\left((a-b)^2 + c^2 - s^2\right)$ is positive, since A and B are both outside the first circle.

In the general case, there are two circles that satisfy the stated conditions, and they are each uniquely determined by the coordinate d.

If the problem asks to perform the construction with only straightedge and compass, then it can be achieved by solving the quadratic equation for d geometrically, via, for example, the Carlyle circle construction, since all coefficients are constructible. Alternatively, the construction can be carried out by following the formula for d, since it only involves operations that can be performed with straightedge and compass. Note that it can be assumed that, for example, $s = 1$, since everything is scale-invariant.

In the case $s^2 = c^2$, the construction can also be performed with straightedge and compass, since the formula for d involves constructible operations only.

Problem 7.22

Prove that if a triangle and a square are circumscribed about the same circle, then the portion of the square contained inside the triangle makes up more than half of the perimeter of the square.
(Note that the triangle is generic: it is not necessarily equilateral).

Solution

To do

Problem 7.23

Let ABC be a triangle. Let M be the midpoint of $\overline{AC}$. Let $\overline{CL}$ be the angle bisector of $\widehat{BCA}$, with $L \in \overline{AB}$. Let P be the intersection point of $\overline{CL}$ and $\overline{BM}$. Prove that $\frac{\overline{CP}}{\overline{PL}} - \frac{\overline{AC}}{\overline{CB}} = 1$.

Solution

To do

Problem 7.24

How many unordered pairs of triangles have as union a given quadrilateral?

Solution

To do

Problem 7.25

n segments are given on the plane. Prove that the number of triangles whose sides are among those segments is $O(n^{3/2})$.

Solution

To do

Problem 7.26

Prove that any convex polygon of area 1 contains a triangle of area $\frac{1}{4}$.

Solution

Let d be the maximum distance between any two vertices of the polygon, and let A and B be two vertices at distance d. The vertices of the polygon, and therefore also the polygon itself, are contained in the strip delimited by the two lines perpendicular to $\overline{AB}$ that pass, respectively, through A and through B. This is because any point outside of that region has distance to A or to B greater than d.

Let C be a vertex of the polygon at maximal distance to the line AB, and let r be that distance. The vertices of the polygon, and therefore also the polygon itself, are contained in the strip delimited by the two lines parallel to $\overline{AB}$ at distance r from the line AB. This is because any point outside of that region has distance to AB greater than r.

Therefore, the polygon is contained in the intersection of the two strips, which is a rectangle of side lengths d and $2r$.

The triangle ABC is contained in the polygon (by convexity), and has area $\frac{1}{2}rd$. The polygon has area less than or equal to $2rd$ (the area of the rectangle containing it). Hence, the triangle's area is at least one fourth of the polygon's area.

Problem 7.27

Let $ABCDE$ be a convex pentagon. The triangles ABC, BCD, CDE, DEA, EAB all have area 1. Determine the area of the pentagon.

Solution

To do

Solid Geometry

Problem 8.1

Prove that if a sphere is tangent to all the edges of a three-dimensional quadrilateral, then the points of tangency are coplanar.

Solution

To do

Problem 8.2

Show that if in a tetrahedron the sums of lengths of opposite edges are all equal, then the sums of opposite dihedral angles are all equal.

Solution

To do

Problem 8.3

Find an equivalent condition for the bisectors of two trihedral angles of a tetrahedron to intersect.
(Note: a bisector of a trihedral angle is the locus of points that are equidistant from its three line edges.)

Solution

To do

Problem 8.4

Determine the tetrahedrons of a given volume that maximise the radius of their inscribed sphere.

Solution

To do

Problem 8.5

Let h_1, h_2, h_3, h_4 be the lengths of the altitudes of a tetrahedron. Let O be an interior point of the tetrahedron. Let d_1, d_2, d_3, d_4 be the distances between O and the planes containing the faces of the tetrahedron. Show that $h_1^4 + h_2^4 + h_3^4 + h_4^4 \geq 2^{10} d_1 d_2 d_3 d_4$.

Solution

To do

Problem 8.6

Prove that the heights of a tetrahedron are concurrent if and only if one of the heights has its base in the orthocenter of the corresponding face.

Solution

To do

Problem 8.7

Prove that in any tetrahedron, the circumradius R and the inradius r are such that $R \geq 3r$.

Solution

To do

Problem 8.8

Given a tetrahedron of unit volume, and one point on each of its sides, cut off corners from each vertex using the given points on the sides exiting that vertex. Prove that at least one of the cutoff parts has volume less than or equal to $\frac{1}{8}$.

Solution

To do

Problem 8.9

Let $ABCD$ be a tetrahedron such that ABC is equilateral and $\widehat{BAD} \cong \widehat{ACD} \cong \widehat{BCD}$. Prove that $ABCD$ is a regular pyramid on the base ABC; that is: prove that $\overline{AD} \cong \overline{BD} \cong \overline{CD}$.

Solution

To do

Problem 8.10

Prove that if the faces of a tetrahedron all have the same area, then they are congruent.

Solution

To do

Problem 8.11

Let $ABCD$ be a tetrahedron. Let O be a point on the face ABC. Prove that:

$$\frac{1}{2}\left(\widehat{ADB}+\widehat{BDC}+\widehat{CDA}\right) < \widehat{ODA}+\widehat{ODB}+\widehat{ODC} < \widehat{ADB}+\widehat{BDC}+\widehat{CDA}$$

Solution

To do

Problem 8.12

Prove that the sum of the measures of all dihedral angles of a tetrahedron is greater than 2π and less than 3π, and that for any value in that range there exists a tetrahedron that achieves it.

Solution

To do

Problem 8.13

If a tetrahedron is contained inside another tetrahedron, then is the sum of the lengths of the sides of the inner one less than that of the outer one? Is the sum of the areas of the faces of the inner tetrahedron less than that of the outer one?

Solution

To do

Problem 8.14

A regular tetrahedron $ABCD$ with side length a has its vertices on the surface of a double-cone whose vertex angle is $\frac{\pi}{2}$. The side $\overline{AB}$ lies on a generator of the cone. Determine the distance from the vertex of the cone to the line CD.

Solution

To do

Problem 8.15

Can a cube be inside a half-cone, with 7 vertices on the surface of the cone?

Solution

To do

Problem 8.16

Determine the distance between a circle inscribed in a face of a cube and a circle circumscribed about an adjacent face of the cube.

Solution

To do

Problem 8.17

Prove that if all the faces of a convex polyhedron are triangles, then there is an edge such that the angles that it forms with its adjacent co-facial edges are all acute.

Solution

To do

Problem 8.18

Prove that an irregular octahedron is completely contained in the union of the balls that have its edges as diameters.

Solution

To do

Problem 8.19

Determine whether it is possible for a planar section of a rectangular parallelepiped to be an equilateral (or regular?) pentagon.

Solution

To do

Problem 8.20

Determine whether for any trihedral angle there exists a plane that intersects it in an equilateral triangle.

Solution

To do

Problem 8.21

Prove that there exists a convex polyhedron of volume 1 that does not contain a tetrahedron of volume $\dfrac{1}{8}$.

Prove that any convex polyhedron of volume 1 contains a tetrahedron of volume $\dfrac{1}{27}$. Try to improve the estimate to a larger constant.

Solution

Consider a sphere of radius r. Its volume is $\frac{4}{3}\pi r^3$. The regular tetrahedron inscribed in the sphere has side length $a = \frac{2}{3}\sqrt{6}r$, and thus its volume is $\frac{\sqrt{2}}{12}a^3 = \frac{8}{27}\sqrt{3}r^3$. The ratio of the volume of the tetrahedron to the volume of the sphere is $\frac{2\sqrt{3}}{9\pi} < \frac{1}{8}$.

Thus, if the inscribed regular tetrahedron is the tetrahedron of highest volume contained in a sphere, then a ratio of volumes greater than or equal to $\frac{1}{8}$ cannot be achieved, for a sphere. Then, considering a polyhedron contained in the sphere that sufficiently approximates the sphere, the same can be said for that polyhedron.

TODO: show that $\frac{2\sqrt{3}}{9\pi} < \frac{1}{8}$; show that the inscribed regular tetrahedron is the tetrahedron of highest volume contained in a sphere; show rigorously that the fact still holds for a "sufficiently spherical" polyhedron.

Now for the second part of the problem.

Let d be the maximum distance between any two vertices of the polyhedron, and let A and B be two vertices at distance d. The vertices of the polyhedron, and therefore also the polyhedron itself, are contained between the two planes orthogonal to $\overline{AB}$ that pass, respectively, through A and through B. This is because any point outside of that region has distance to A or to B greater than d.

Let C be a vertex of the polyhedron at maximal distance to the line AB, and let r be that distance. Let α be the plane on which A, B and C lie. The vertices of the polyhedron, and therefore also the polyhedron itself, are contained between the two planes at distance r from the line AB that are perpendicular to α and to the previous pair of planes. This is because any point outside of that region has distance to AB greater than r.

Let D be a vertex of the polyhedron at maximal distance to the plane α, and let s be that distance. The vertices of the polyhedron, and therefore also the polyhedron itself, are contained between the two planes parallel to α at distance s from α. This is because any point outside of that region has distance to α greater than s.

Therefore, the polyhedron is contained in the intersection of the three regions, which is a rectangular parallelepiped of side lengths d, $2r$ and $2s$.

The tetrahedron $ABCD$ is contained in the polyhedron (by convexity), and has volume $\frac{1}{6}rsd$. The polyhedron has volume less than or equal to $4rsd$ (the volume of the parallelepiped containing it). Hence, the tetrahedron's volume is at least $\frac{1}{24}$ times the polyhedron's volume.

(This greedy argument suggests that in dimension n, the fraction is at least $\frac{1}{n!\cdot 2^{n-1}}$).

Geometric constructions

Problem 9.1

Let ABC be a triangle. Using only straightedge and compass, construct a point $P \in \overline{AB}$ and a point $Q \in \overline{BC}$ such that $\overline{AP} \cong \overline{PQ} \cong \overline{QC}$.

Solution

To do

Problem 9.2

Using only straightedge and compass, construct a quadrilateral, given its angles, in order, and its diagonals, in order.

Solution

To do

Problem 9.3

Using only straightedge and compass, reconstruct a quadrilateral, given segments congruent to its four sides, in order, and a segment congruent to the segment between the midpoints of the first and third sides.

Solution

To do

Problem 9.4

Given a point and an angle on a plane, construct, using only straightedge and compass, a line through the point that cuts the angle into a triangle of minimum perimeter.
Additionally, given also a segment, construct a line through the point that cuts the angle into a triangle whose perimeter is the length of the segment.

Solution

To do

Problem 9.5

Given a circle and one of its diameters, and given a point on the plane that does not lie on the circle nor on the line containing the diameter, construct, using only a straightedge, the perpendicular from the given point to the given diameter.

Solution

To do

Problem 9.6

Given a circle and one of its diameters, and given a point on the circle, distinct from the endpoints of the diameter, construct, using only a straightedge, the perpendicular from the given point to the given diameter.

Solution

To do

Problem 9.7

Given a segment and a positive integer n, divide the segment into n parts of equal lengths, using only a compass.

Solution

To do

Problem 9.8

Given two parallel segments and a positive integer n, divide one of the segments into n parts of equal lengths, using only a straightedge.

Solution

To do

Problem 9.9

Given two segments on the plane, of lengths a and b, construct, using only straightedge and compass, a segment of length c such that:

$$\sqrt[4]{c} = \sqrt[4]{a} + \sqrt[4]{b}$$

Note: a segment of length 1 is not given.

Solution

Recall that given two segments, of lengths r and s, a segment whose length is the geometric mean of r and s can be constructed with compass and straightedge, via the semicircle construction (which does not require a segment of length 1).
Hence it is possible to construct a segment of length $\sqrt{ab}$, and then also segments of lengths $\sqrt{a\sqrt{ab}} = \sqrt[4]{a^3b}$ and $\sqrt{b\sqrt{ab}} = \sqrt[4]{ab^3}$. Finally, a segment of the required length c can be constructed by combining the previous segments, since:

$$c = \left(\sqrt[4]{a} + \sqrt[4]{b}\right)^4 = a + 4\sqrt[4]{a^3b} + 6\sqrt{ab} + 4\sqrt[4]{ab^3} + b$$

Problem 9.10

Reconstruct a square given one point from each side, using only straightedge and compass.

Solution

To do

Problem 9.11

Using only straightedge and compass, construct the directrix and focus of a given parabola.

Solution

To do

Problem 9.12

Using only straightedge and compass, find the diameter of a given sphere.

Solution

To do

Other

Problem 10.1

Let R, S, T be sets, with $\#(R) \geq 2$, $\#(S) \geq 2$, and $\#(T) \geq 3$. Let $f : R \times S \to T$ be such that $\#(\mathrm{im}(f)) \geq 3$. The elements $a \in R$ and $b \in S$ are such that the functions $S \to T : y \mapsto f(a, y)$ and $R \to T : x \mapsto f(x, b)$ are not constant.
Prove that there exist $p, r \in R$ and $q, s \in S$ such that $f(p, q)$, $f(r, q)$, $f(p, s)$ are all distinct.

Solution

Since the function $S \to T : y \mapsto f(a, y)$ is not constant, there exists $c \in S$ such that $f(a, c) \neq f(a, b)$. And since the function $R \to T : x \mapsto f(x, b)$ is not constant, there exists $d \in R$ such that $f(d, b) \neq f(a, b)$.

Now, if c and d can be chosen in such a way that $f(b,d) \neq f(a,c)$, then $p = a$, $q = b$, $r = d$, $s = t$ satisfy the requirements.
Otherwise, it means that for every $(x,y) \in R \times S \setminus \{(a,b)\}$ $f(x,b) = f(a,y) = v$ (for some $v \in T$). Since $\#(\text{im}(f)) \geq 3$, there exists $(r,q) \in R \times S$ such that $f(r,q)$, v, and $f(a,b)$ are all distinct. Necessarily, then, $r \neq a$ and $q \neq b$. Now, the requirements are satisfied by taking $(p,s) = (a,b)$.

Problem 10.2

Let $f : \mathbb{R} \to \mathbb{R}$ be a function, with $f(0) = 0$, $f(1) = 1$, $f(88) = \sqrt{2}$. Prove that there exist $x, y \in \mathbb{R}$ with $|x - y| \leq 4$ such that $f(x+1) > f(x)$ and $f(y + 2^n) \neq f(y)$ for some $n \in \mathbb{N}$.

Solution

To do