

# Coffin problems

## Problems

### Evaluations

#### Problem 1.1

Let  $a_1, \dots, a_n \in \mathbb{C}$  be an arithmetic progression, with  $\cos(a_i) \neq 0$  for all  $1 \leq i \leq n$ . Evaluate the sum:

$$\sum_{i=1}^{n-1} \frac{1}{\cos(a_i) \cos(a_{i+1})}$$

#### Problem 1.2

Find  $\sin(1^\circ)$ .

#### Problem 1.3

Evaluate  $\tan\left(\frac{1}{7}\pi\right) \cdot \tan\left(\frac{3}{7}\pi\right) \cdot \tan\left(\frac{5}{7}\pi\right)$ .

### Problem 1.4

Determine the number of digits of  $125^{100}$  in base 10.

## Comparisons

### Problem 2.1

Determine which among  $\log_2(3)$  and  $\log_3(5)$  is largest.

### Problem 2.2

Determine which among  $\prod_{n=2}^{40} \log_3(2n)$  and  $2 \prod_{n=2}^{40} \log_3(2n-1)$  is largest.

### Problem 2.3

Determine which among  $\frac{8}{27}\pi$  and  $\sin\left(\frac{8}{7}\right)$  is largest.

### Problem 2.4

Determine which among  $\sqrt[3]{413}$  and  $6 + \sqrt[3]{3}$  is largest.

### Problem 2.5

Prove that:

$$\sqrt[3]{3 + \sqrt[3]{3}} + \sqrt[3]{3 - \sqrt[3]{3}} < 2\sqrt[3]{3}$$

### Problem 2.6

Prove that:

$$\sqrt{3 + 32 \sin^2(15^\circ)} + \cos(22^\circ) + \cos(70^\circ) + \cos(88^\circ) + 2\sqrt{2} \sin(15^\circ) > \frac{3}{2} (\cos(11^\circ) + \cos(35^\circ) + \cos(44^\circ))^2$$

## Equations

### Problem 3.1

For  $a \in \mathbb{R}$ , find all real numbers  $x \geq -a$  that satisfy:

$$\sqrt{a + \sqrt{a + x}} = x$$

### Problem 3.2

Find all  $x \in \mathbb{R}$  that satisfy:

$$2\sqrt[3]{2x-1} = x^3 + 1$$

### Problem 3.3

Let  $a \in \mathbb{R}^+$ . Find all  $x \in \mathbb{R}^+$  that satisfy:

$$x^{x^a} = a$$

### Problem 3.4

Find all  $x \in \mathbb{R}$  (or  $\mathbb{C}$ ?) that satisfy:

$$x^4 - 14x^3 + 66x^2 - 115x + 66 + \frac{1}{4} = 0$$

### Problem 3.5

Find all  $x, y \in \mathbb{R}$  that satisfy:

$$\begin{cases} y \cdot (x + y)^2 = 9 \\ y \cdot (x^3 - y^3) = 7 \end{cases}$$

### Problem 3.6

For each  $n \in \mathbb{N}$ , determine the set  $M_n$  of pairs  $(a, b) \in \mathbb{R}^2$  such that the equation  $x^2 - a = |x - b|$  has exactly  $n$  solutions in  $\mathbb{R}$ .

Describe the plot of each set  $M_n$  in  $\mathbb{R}^2$ .

### Problem 3.7

Investigate the following equation, for  $x \in \mathbb{R}$ , parametrised by  $a \in \mathbb{R}$ :

$$4^x + 2 = 2^x a \sin(\pi x)$$

### Problem 3.8

Find all  $x \in \mathbb{R}$  that satisfy:

$$\sin^7(x) + \frac{1}{\sin^3(x)} = \cos^7(x) + \frac{1}{\cos^3(x)}$$

### Problem 3.9

Find all  $x \in \mathbb{R}$  that satisfy:

$$\sin^{\frac{11}{7}}(x) + \cos^{\frac{19}{11}}(x) = \sqrt{\frac{19}{7}}$$

### Problem 3.10

Find all  $x \in \mathbb{R}$  that satisfy:

$$\left(1 - \frac{1}{8} \cos^2(x)\right)^8 = \sin^2(x)$$

### Problem 3.11

Find all  $x \in (0, \pi)$  that satisfy:

$$\cot(x) = \sin\left(x + \frac{\pi}{4}\right)$$

### Problem 3.12

Find all  $x \in \mathbb{R}$  that satisfy:

$$\sin^3(x) \cos\left(\frac{x}{2}\right) + \frac{1}{2} \sin(x) \sin\left(\frac{x}{2}\right) \left(1 + 2 \cos\left(\frac{x}{2}\right)\right) - 6 \sin^2\left(\frac{x}{2}\right) - 1 = 0$$

### Problem 3.13

Find all  $x \in \mathbb{R}^+$  that satisfy:

$$\frac{1}{16^x} = \log_{\frac{1}{16}}(x)$$

## Inequalities

### Problem 4.1

Find all  $x \in [-1, 1]$  that satisfy:

$$x \cdot (8\sqrt{1-x} + \sqrt{1+x}) \leq 11\sqrt{1+x} - 16\sqrt{1-x}$$

### Problem 4.2

Find all  $a \in \mathbb{R}$  such that for any  $x \in \mathbb{R}^+$  the following holds:

$$ax^2 + 2x > 3a - 1$$

### Problem 4.3

Find all  $x \in \mathbb{R}$  that satisfy:

$$2^{\sin(x)} + 2^{\cos(x)} \geq 2^{1-\frac{1}{\sqrt{2}}}$$

### Problem 4.4

Show that for all  $x \in (0, \frac{\pi}{2})$  the following holds:

$$\frac{1}{\sin^2(x)} < \frac{1}{x^2} + 1 - \frac{4}{\pi^2}$$

### Problem 4.5

Show that for all  $x \in (0, \frac{\pi}{2}]$  the following holds:

$$\frac{\sin^3(x)}{x^3} > \cos(x)$$

### Problem 4.6

Find all  $(x, y) \in (-3, 3) \times \mathbb{R}$  that satisfy:

$$3^y \log_3(9 - x^2) \leq 1 + 3^{2y}$$

## Algebra and Number Theory

### Problem 5.1

Prove that the set  $\{n + m\sqrt{2} \mid n, m \in \mathbb{Z}\}$  is dense in  $\mathbb{R}$ .

### Problem 5.2

For each  $a \in \mathbb{N}$ , let  $P(a)$  be the set of prime divisors of  $a$ . Let:

$$S = \{(a, b) \in \mathbb{N}^2 \mid a \neq b, P(a) = P(b), P(a+1) = P(b+1)\}$$

Determine whether  $S$  is finite or infinite.

### Problem 5.3

For each  $i \in \mathbb{Z}^+$ , let  $p_i$  be the  $i$ -th prime number. Prove that for every  $n \in \mathbb{Z}^+$ :

$$\prod_{i=1}^n p_i < 4^{p_n}$$

### Problem 5.4

Prove that  $\sin(10^\circ)$  is irrational.

### Problem 5.5

Do there exist irrational numbers  $a, b \in \mathbb{R}^+$  such that  $a^b$  is rational?

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### Problem 5.6

The digit expansion of a number  $a \in (0, 1)$  has 0 as first digit, then for every  $n \in \mathbb{N}$ , the digits  $(2^n + 1)$ -th to  $2^{n+1}$ -th are the opposite of the digits 1-st to  $2^n$ -th, respectively, where the opposite of the digit 1 is the digit 0, and viceversa. Prove that  $a$  is irrational.

### Problem 5.7

Does there exist a (non-degenerate) equilateral triangle in  $\mathbb{R}^2$  whose vertices are rational points? (That is, whose vertices are in  $\mathbb{Q}^2$ ).

### Problem 5.8

Does there exist a point inside a square of side length 1, such that its distances to the four vertices of the square are all rational?

### Problem 5.9

Does there exist a set  $S \subseteq \mathbb{R}^2$  of six points, no three of which are collinear, and such that for every  $p, q \in S$  the distance  $d(p, q)$  is integer?

**Problem 5.10**

Determine all pairs of positive integers  $(a, b)$  such that  $a^b = b^a$ .

**Problem 5.11**

Do there exist infinitely many pairs  $(a, b) \in \mathbb{N}^2$  such that  $a^2 + (a + 1)^2 = b^2$  ?

**Problem 5.12**

Determine all  $a, b, c, d \in \mathbb{Q}$  such that

$$(a + b\sqrt{2})^2 + (c + d\sqrt{2})^2 = 5 + 4\sqrt{2}$$

**Problem 5.13**

Rationalise the denominator in the following fraction:

$$\frac{1}{\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}}$$

**Analysis****Problem 6.1**

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a monotonically increasing function. Let  $a, b \in \mathbb{R}$  with  $a < b$ . Determine which points  $c \in (a, b)$  minimize the value:

$$\int_a^c f(x) - f(a) \, dx + \int_c^b f(b) - f(x) \, dx$$

**Problem 6.2**

Let  $f : [0, 1] \rightarrow \mathbb{R}^+$  be a continuous function. Let  $a, b \in \mathbb{R}^+$  such that  $a \leq f \leq b$ . Prove that:

$$ab \int_0^1 \frac{1}{f(x)} \, dx \leq a + b - \int_0^1 f(x) \, dx$$

### Problem 6.3

Find all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that for any  $x, y \in \mathbb{R}$  the following holds:

$$f(x) - f(y) \leq (x - y)^2$$

### Problem 6.4

Find all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that for any  $x \in \mathbb{R}$  the following holds:

$$f(f(x)) = x^2 - 2$$

### Problem 6.5

Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{R}$ . If  $\lim_{n \rightarrow +\infty} (a_{n+1} - a_n) = 0$ , then does  $\lim_{n \rightarrow +\infty} a_n$  exist (finite or infinite)?

### Problem 6.6

Prove that:

$$\sum_{n=1}^{1000} \frac{1}{n^3 + 3n^2 + 2n} < \frac{1}{4}$$

### Problem 6.7

Prove that if  $a, b, c, d \in \mathbb{R}$  are such that  $a^2 + 4b^2 = 4$  and  $cd = 4$ , then  $(a - d)^2 + (b - c)^2 \geq \frac{8}{5}$ .

## Plane Geometry

### Problem 7.1

Let  $ABC$  be a triangle, with  $\widehat{ABC} = 80^\circ$ . Let  $O$  be a point inside  $ABC$  such that  $\widehat{OAC} = 10^\circ$  and  $\widehat{OCA} = 30^\circ$ .

Express the angle  $\widehat{ABO}$  in terms of  $\frac{OB}{AC}$ .

### Problem 7.2

Prove that if the lengths of the angle bisectors of a triangle are all less than or equal to 1, then the area of the triangle is less than or equal to  $\frac{\sqrt{3}}{3}$ .

### Problem 7.3

Four circles on a plane are mutually tangent to each other. The points of tangency are all distinct. Three of the circles have collinear centers. Determine the distance between the center of fourth circle and the line through the centers of the others, in terms of the radius of the fourth circle.

(There are two cases: one for internal tangency and one for external tangency)

### Problem 7.4

Let  $ABC$  be a triangle. Let  $\gamma$  be its circumcircle. Let  $\alpha_1, \alpha_2, \alpha_3$  be circles such that  $\alpha_1$  is tangent to  $\overline{BC}$ , to  $\overline{CA}$ , and to  $\gamma$  (internally);  $\alpha_2$  is tangent to  $\overline{AB}$ , to  $\overline{CA}$ , and to  $\gamma$  (internally);  $\alpha_3$  is tangent to  $\overline{AB}$ , to  $\overline{BC}$ , and to  $\gamma$  (internally). Determine the radius of  $\gamma$ , given the radii of  $\alpha_1, \alpha_2, \alpha_3$ .

Alternative formulation.

Three circles are each tangent to a (distinct) unordered pair of (distinct) sides of a triangle and to the circumcircle of the triangle (internally). Determine the radius of the circumcircle, given the radii of the three circles.

### Problem 7.5

Let  $ABC$  be an equilateral triangle, and let  $O$  be a point inside it. Show that the lengths  $\overline{AO}, \overline{BO}, \overline{CO}$  can be the side lengths of a triangle, and determine the measures of the internal angles in such triangle, in terms of  $\widehat{AOB}$  and  $\widehat{BOC}$ .

### Problem 7.6

Prove that a quadrilateral  $ABCD$  is a rhombus if and only if the triangles  $AOB, BOC, COD, DOA$  are isoperimetric, where  $O$  is the intersection of the diagonal lines  $AC$  and  $BD$ .

### Problem 7.7

Given a triangle, let  $R$  be the radius of its circumscribed circle, and  $r$  the radius of its inscribed circle. Determine the distance  $s$  between the centers of these two circles.

Determine the set of possible values of  $s$  among all triangles that have a fixed circumradius  $R$ ; when are the extremes achieved?

### Problem 7.8

Given two intersecting lines  $r, s$  on a plane, and a real number  $a \geq 0$ , find the locus of points  $P$  of the plane such that  $d(P, r) + d(P, s) = a$ .

### Problem 7.9

Prove that the area of a quadrilateral with side lengths  $a, b, c, d$  which admits both inscribed and circumscribed circles is  $\sqrt{abcd}$ .

### Problem 7.10

Determine the shortest networks that connect the four vertices of a square to each other.

### Problem 7.11

Let  $U = \{A, B, C\}$  be a partition of  $\mathbb{R}^2$ . Prove that for any  $a \in \mathbb{R}^+$  there exists  $S \in U$  such that there exist  $p, q \in S$  with  $d(p, q) = a$ .

### Problem 7.12

Prove that in a cyclic quadrilateral, the perpendiculars to each side passing through the midpoint of the opposite side are concurrent.

### Problem 7.13

Determine the quadrilateral with the largest area, given the lengths of its four sides, in order.

### Problem 7.14

Let  $\overline{AB}$  be a chord in a circle, and let  $M$  be its midpoint. Let  $\overline{CD}$  and  $\overline{EF}$  be two other chords in the circle that pass through the point  $M$ , with  $C$  and  $F$  on opposite sides of  $\overline{AB}$ . Prove that  $\overline{CF}$  intersects  $\overline{AB}$  at a point  $P$ , and  $\overline{ED}$  intersects  $\overline{AB}$  at a point  $Q$ , on opposite sides of  $M$ , such that  $\overline{MP} \cong \overline{MQ}$ .

**Problem 7.15**

Given a triangle, determine a line that halves both its area and its perimeter.

**Problem 7.16**

Let  $ABCD$  be a trapezoid with bases  $\overline{AB}$  and  $\overline{CD}$ . Given a point  $P \in \overline{AB}$ , determine two points  $Q_1, Q_2 \in \overline{CD}$  that, respectively, maximize the area of the quadrilateral intersection of the triangles  $ABQ_1$  and  $CDP$ , and minimize the area of the quadrilateral intersection of the triangles  $ABQ_2$  and  $CDP$ .

**Problem 7.17**

Let  $a, b, c$  be the side lengths of a triangle, and let  $\alpha, \beta, \gamma$  be the measures of their opposite angles, respectively. Prove that:

$$\frac{b+c-2a}{\sin(\frac{\alpha}{2})} + \frac{c+a-2b}{\sin(\frac{\beta}{2})} + \frac{a+b-2c}{\sin(\frac{\gamma}{2})} \geq 0$$

**Problem 7.18**

Let  $\alpha, \beta, \gamma$  be the measures of the internal angles of a triangle. Prove that:

$$\sqrt{\sin(\alpha)} + \sqrt{\sin(\beta)} + \sqrt{\sin(\gamma)} \leq \frac{9}{\sqrt{10}}$$

**Problem 7.19**

Let  $a, b, c$  be the side lengths of a triangle, and let  $\alpha, \beta, \gamma$  be the measures of their opposite angles, respectively. Prove that:

$$\frac{\pi}{3} \leq \frac{a \cdot \alpha + b \cdot \beta + c \cdot \gamma}{a + b + c} \leq \frac{\pi}{2}$$

**Problem 7.20**

Let  $P$  be a point in an equilateral (or regular?) hexagon of side length 1. Prove that the sum of the distances between  $P$  and the vertices of the hexagon is less than or equal to  $4 + 2\sqrt{3}$ .

### Problem 7.21

A circle is given on a plane. Given two points on the plane, outside the circle, construct a circle that passes through those two points and is tangent to the first circle.

(Is the problem asking for a straightedge and compass construction?)

### Problem 7.22

Prove that if a triangle and a square are circumscribed about the same circle, then the portion of the square contained inside the triangle makes up more than half of the perimeter of the square.

(Note that the triangle is generic: it is not necessarily equilateral).

### Problem 7.23

Let  $ABC$  be a triangle. Let  $M$  be the midpoint of  $\overline{AC}$ . Let  $\overline{CL}$  be the angle bisector of  $\widehat{BCA}$ , with  $L \in \overline{AB}$ . Let  $P$  be the intersection point of  $\overline{CL}$  and  $\overline{BM}$ . Prove that  $\frac{CP}{PL} - \frac{AC}{CB} = 1$ .

### Problem 7.24

How many unordered pairs of triangles have as union a given quadrilateral?

### Problem 7.25

$n$  segments are given on the plane. Prove that the number of triangles whose sides are among those segments is  $O(n^{3/2})$ .

### Problem 7.26

Prove that any convex polygon of area 1 contains a triangle of area  $\frac{1}{4}$ .

### Problem 7.27

Let  $ABCDE$  be a convex pentagon. The triangles  $ABC$ ,  $BCD$ ,  $CDE$ ,  $DEA$ ,  $EAB$  all have area 1. Determine the area of the pentagon.

## Solid Geometry

### Problem 8.1

Prove that if a sphere is tangent to all the edges of a three-dimensional quadrilateral, then the points of tangency are coplanar.

### Problem 8.2

Show that if in a tetrahedron the sums of lengths of opposite edges are all equal, then the sums of opposite dihedral angles are all equal.

### Problem 8.3

Find an equivalent condition for the bisectors of two trihedral angles of a tetrahedron to intersect. (Note: a bisector of a trihedral angle is the locus of points that are equidistant from its three line edges.)

### Problem 8.4

Determine the tetrahedrons of a given volume that maximise the radius of their inscribed sphere.

### Problem 8.5

Let  $h_1, h_2, h_3, h_4$  be the lengths of the altitudes of a tetrahedron. Let  $O$  be an interior point of the tetrahedron. Let  $d_1, d_2, d_3, d_4$  be the distances between  $O$  and the planes containing the faces of the tetrahedron. Show that  $h_1^4 + h_2^4 + h_3^4 + h_4^4 \geq 2^{10} d_1 d_2 d_3 d_4$ .

### Problem 8.6

Prove that the heights of a tetrahedron are concurrent if and only if one of the heights has its base in the orthocenter of the corresponding face.

### Problem 8.7

Prove that in any tetrahedron, the circumradius  $R$  and the inradius  $r$  are such that  $R \geq 3r$ .

### Problem 8.8

Given a tetrahedron of unit volume, and one point on each of its sides, cut off corners from each vertex using the given points on the sides exiting that vertex. Prove that at least one of the cutoff parts has volume less than or equal to  $\frac{1}{8}$ .

### Problem 8.9

Let  $ABCD$  be a tetrahedron such that  $ABC$  is equilateral and  $\widehat{BAD} \cong \widehat{ACD} \cong \widehat{BCD}$ . Prove that  $ABCD$  is a regular pyramid on the base  $ABC$ ; that is: prove that  $\overline{AD} \cong \overline{BD} \cong \overline{CD}$ .

### Problem 8.10

Prove that if the faces of a tetrahedron all have the same area, then they are congruent.

### Problem 8.11

Let  $ABCD$  be a tetrahedron. Let  $O$  be a point on the face  $ABC$ . Prove that:

$$\frac{1}{2}(\widehat{ADB} + \widehat{BDC} + \widehat{CDA}) < \widehat{ODA} + \widehat{ODB} + \widehat{ODC} < \widehat{ADB} + \widehat{BDC} + \widehat{CDA}$$

### Problem 8.12

Prove that the sum of the measures of all dihedral angles of a tetrahedron is greater than  $2\pi$  and less than  $3\pi$ , and that for any value in that range there exists a tetrahedron that achieves it.

### Problem 8.13

If a tetrahedron is contained inside another tetrahedron, then is the sum of the lengths of the sides of the inner one less than that of the outer one? Is the sum of the areas of the faces of the inner tetrahedron less than that of the outer one?

### Problem 8.14

A regular tetrahedron  $ABCD$  with side length  $a$  has its vertices on the surface of a double-cone whose vertex angle is  $\frac{\pi}{2}$ . The side  $\overline{AB}$  lies on a generator of the cone. Determine the distance from the vertex of the cone to the line  $CD$ .

**Problem 8.15**

Can a cube be inside a half-cone, with 7 vertices on the surface of the cone?

**Problem 8.16**

Determine the distance between a circle inscribed in a face of a cube and a circle circumscribed about an adjacent face of the cube.

**Problem 8.17**

Prove that if all the faces of a convex polyhedron are triangles, then there is an edge such that the angles that it forms with its adjacent co-facial edges are all acute.

**Problem 8.18**

Prove that an irregular octahedron is completely contained in the union of the balls that have its edges as diameters.

**Problem 8.19**

Determine whether it is possible for a planar section of a rectangular parallelepiped to be an equilateral (or regular?) pentagon.

**Problem 8.20**

Determine whether for any trihedral angle there exists a plane that intersects it in an equilateral triangle.

**Problem 8.21**

Prove that there exists a convex polyhedron of volume 1 that does not contain a tetrahedron of volume  $\frac{1}{8}$ .

Prove that any convex polyhedron of volume 1 contains a tetrahedron of volume  $\frac{1}{27}$ . Try to improve the estimate to a larger constant.

## Geometric constructions

### Problem 9.1

Let  $ABC$  be a triangle. Using only straightedge and compass, construct a point  $P \in \overline{AB}$  and a point  $Q \in \overline{BC}$  such that  $\overline{AP} \cong \overline{PQ} \cong \overline{QC}$ .

### Problem 9.2

Using only straightedge and compass, construct a quadrilateral, given its angles, in order, and its diagonals, in order.

### Problem 9.3

Using only straightedge and compass, reconstruct a quadrilateral, given segments congruent to its four sides, in order, and a segment congruent to the segment between the midpoints of the first and third sides.

### Problem 9.4

Given a point and an angle on a plane, construct, using only straightedge and compass, a line through the point that cuts the angle into a triangle of minimum perimeter.

Additionally, given also a segment, construct a line through the point that cuts the angle into a triangle whose perimeter is the length of the segment.

### Problem 9.5

Given a circle and one of its diameters, and given a point on the plane that does not lie on the circle nor on the line containing the diameter, construct, using only a straightedge, the perpendicular from the given point to the given diameter.

### Problem 9.6

Given a circle and one of its diameters, and given a point on the circle, distinct from the endpoints of the diameter, construct, using only a straightedge, the perpendicular from the given point to the given diameter.

**Problem 9.7**

Given a segment and a positive integer  $n$ , divide the segment into  $n$  parts of equal lengths, using only a compass.

**Problem 9.8**

Given two parallel segments and a positive integer  $n$ , divide one of the segments into  $n$  parts of equal lengths, using only a straightedge.

**Problem 9.9**

Given two segments on the plane, of lengths  $a$  and  $b$ , construct, using only straightedge and compass, a segment of length  $c$  such that:

$$\sqrt[4]{c} = \sqrt[4]{a} + \sqrt[4]{b}$$

Note: a segment of length 1 is not given.

**Problem 9.10**

Reconstruct a square given one point from each side, using only straightedge and compass.

**Problem 9.11**

Using only straightedge and compass, construct the directrix and focus of a given parabola.

**Problem 9.12**

Using only straightedge and compass, find the diameter of a given sphere.

## Other

### Problem 10.1

Let  $R, S, T$  be sets, with  $\#(R) \geq 2$ ,  $\#(S) \geq 2$ , and  $\#(T) \geq 3$ . Let  $f : R \times S \rightarrow T$  be such that  $\#(\text{im}(f)) \geq 3$ . The elements  $a \in R$  and  $b \in S$  are such that the functions  $S \rightarrow T : y \mapsto f(a, y)$  and  $R \rightarrow T : x \mapsto f(x, b)$  are not constant.

Prove that there exist  $p, r \in R$  and  $q, s \in S$  such that  $f(p, q), f(r, q), f(p, s)$  are all distinct.

### Problem 10.2

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function, with  $f(0) = 0$ ,  $f(1) = 1$ ,  $f(88) = \sqrt{2}$ . Prove that there exist  $x, y \in \mathbb{R}$  with  $|x - y| \leq 4$  such that  $f(x + 1) > f(x)$  and  $f(y + 2^n) \neq f(y)$  for some  $n \in \mathbb{N}$ .