Coffin problems

Solutions

Evaluations

Problem 1.1

Let $a, r \in \mathbb{C}$. For $n \in \mathbb{N}$, if $\cos(a+kr) \neq 0$ for all integers $0 \leq k \leq n$, evaluate the sum:

$$\sum_{k=0}^{n-1} \frac{1}{\cos(a+kr)\cos(a+(k+1)r)}$$

Solution 1

For r not a multiple of π:

$$\sum_{k=0}^{n-1} \frac{1}{\cos(a+kr)\cos(a+(k+1)r)} = \frac{\sin(nr)}{\sin(r)\cos(a)\cos(a+nr)} = \frac{\tan(a+nr)-\tan(a)}{\sin(r)}$$

In fact:

$$\begin{aligned}
\sum_{k=0}^{n-1} \frac{\sin(r)}{\cos(a+kr)\cos(a+(k+1)r)} &= \sum_{k=0}^{n-1} \frac{\sin\Big((a+(k+1)r)-(a+kr)\Big)}{\cos(a+kr)\cos(a+(k+1)r)} = \\
&= \sum_{k=0}^{n-1} \frac{\sin\Big(a+(k+1)r\Big)\cos(a+kr)-\sin(a+kr)\cos\Big(a+(k+1)r\Big)}{\cos(a+kr)\cos(a+(k+1)r)} = \\
&= \sum_{k=0}^{n-1} \tan\Big(a+(k+1)r\Big)-\tan(a+kr) = \\
&= \tan(a+nr)-\tan(a)
\end{aligned}$$

Solution 2

For r not a multiple of π:

$$\sum_{k=0}^{n-1} \frac{1}{\cos(a+kr)\cos(a+(k+1)r)} = \frac{\sin(nr)}{\sin(r)\cos(a)\cos(a+nr)} = \frac{\tan(a+nr)-\tan(a)}{\sin(r)}$$

This can be proven by induction on n. For $n=0$ the sum is 0, so it is trivial. Then:

$$\begin{aligned}
\sum_{k=0}^{n} \frac{1}{\cos(a+kr)\cos(a+(k+1)r)} &= \frac{1}{\cos(a+nr)\cos(a+nr+r)} + \sum_{k=0}^{n-1} \frac{1}{\cos(a+kr)\cos(a+(k+1)r)} = \\
&= \frac{1}{\cos(a+nr)\cos(a+nr+r)} + \frac{\sin(nr)}{\sin(r)\cos(a)\cos(a+nr)} = \\
&= \frac{\frac{\sin(nr)}{\sin(r)}\cos(a+nr+r)+\cos(a)}{\cos(a)\cos(a+nr)\cos(a+nr+r)} = \\
&= \frac{\frac{\sin(nr)}{\sin(r)}\cos(a+nr+r)+\cos(a+nr-nr)}{\cos(a)\cos(a+nr)\cos(a+nr+r)} = \\
&= \frac{\frac{\sin(nr)}{\sin(r)}(\cos(a+nr)\cos(r)-\sin(a+nr)\sin(r))+\cos(a+nr)\cos(nr)+\sin(a+nr)\sin(nr)}{\cos(a)\cos(a+nr)\cos(a+nr+r)} = \\
&= \frac{\cos(a+nr)\Big(\frac{\sin(nr)}{\sin(r)}\cos(r)+\cos(nr)\Big)+\sin(a+nr)\Big(\sin(nr)-\frac{\sin(nr)}{\sin(r)}\sin(r)\Big)}{\cos(a)\cos(a+nr)\cos(a+nr+r)} = \\
&= \frac{\cos(a+nr)\frac{\sin(nr)\cos(r)+\sin(r)\cos(nr)}{\sin(r)}+\sin(a+nr)\cdot 0}{\cos(a)\cos(a+nr)\cos(a+nr+r)} = \\
&= \frac{\sin(nr)\cos(r)+\sin(r)\cos(nr)}{\sin(r)\cos(a)\cos(a+nr+r)} = \frac{\sin((n+1)r)}{\sin(r)\cos(a)\cos(a+(n+1)r)}
\end{aligned}$$

Problem 1.2

Find $\sin(1°)$, $\cos(1°)$, $\tan(1°)$, and $\sin(2°)$, $\cos(2°)$, $\tan(2°)$.

Solution 1

Observe that $1° = 10° - 9°$, and $10° = \frac{\pi}{18}$. Hence:

$$\begin{aligned} \sin(1°) = \sin(10° - 9°) &= \sin(10°)\cos(9°) - \cos(10°)\sin(9°) = \\ &= \frac{1}{2i}(e^{i\frac{\pi}{18}} - e^{-i\frac{\pi}{18}})\cos(9°) - \frac{1}{2}(e^{i\frac{\pi}{18}} + e^{-i\frac{\pi}{18}})\sin(9°) = \\ &= \frac{1}{2}e^{i\frac{\pi}{18}}(-\sin(9°) - i\cos(9°)) + \frac{1}{2}e^{-i\frac{\pi}{18}}(-\sin(9°) + i\cos(9°)) \end{aligned}$$

and:

$$\begin{aligned} \cos(1°) = \cos(10° - 9°) &= \cos(10°)\cos(9°) + \sin(10°)\sin(9°) = \\ &= \frac{1}{2}(e^{i\frac{\pi}{18}} + e^{-i\frac{\pi}{18}})\cos(9°) + \frac{1}{2i}(e^{i\frac{\pi}{18}} - e^{-i\frac{\pi}{18}})\sin(9°) = \\ &= \frac{1}{2}e^{i\frac{\pi}{18}}(\cos(9°) - i\sin(9°)) + \frac{1}{2}e^{-i\frac{\pi}{18}}(\cos(9°) + i\sin(9°)) \end{aligned}$$

The term $e^{i\frac{\pi}{18}}$ is the principal cube root of $e^{i\frac{\pi}{6}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$, and the term $e^{-i\frac{\pi}{18}}$ is the principal cube root of $e^{-i\frac{\pi}{6}} = \frac{\sqrt{3}}{2} - \frac{1}{2}i$, so they can be written as:

$$e^{i\frac{\pi}{18}} = \sqrt[3]{\frac{\sqrt{3}+i}{2}}, \qquad e^{-i\frac{\pi}{18}} = \sqrt[3]{\frac{\sqrt{3}-i}{2}}$$

The values of $\sin(9°)$ and $\cos(9°)$ are the following (they can be calculated using the angle difference formulas from $9° = 45° - 36°$):

$$\begin{aligned} \sin(9°) &= \frac{1}{8}\sqrt{2}\left(\sqrt{5}+1\right) - \frac{1}{4}\sqrt{5-\sqrt{5}} \\ \cos(9°) &= \frac{1}{8}\sqrt{2}\left(\sqrt{5}+1\right) + \frac{1}{4}\sqrt{5-\sqrt{5}} \end{aligned}$$

Altogether:

$$\begin{aligned} \sin(1°) = &\frac{1}{8}\sqrt[3]{\frac{\sqrt{3}+i}{2}} \cdot \left(-\frac{1}{2}\sqrt{2}\left(\sqrt{5}+1\right) + \sqrt{5-\sqrt{5}} - i\left(\frac{1}{2}\sqrt{2}\left(\sqrt{5}+1\right) + \sqrt{5-\sqrt{5}}\right)\right) + \\ &+ \frac{1}{8}\sqrt[3]{\frac{\sqrt{3}-i}{2}} \cdot \left(-\frac{1}{2}\sqrt{2}\left(\sqrt{5}+1\right) + \sqrt{5-\sqrt{5}} + i\left(\frac{1}{2}\sqrt{2}\left(\sqrt{5}+1\right) + \sqrt{5-\sqrt{5}}\right)\right) \end{aligned}$$

and:

$$\begin{aligned} \cos(1°) = &\frac{1}{8}\sqrt[3]{\frac{\sqrt{3}+i}{2}} \cdot \left(\frac{1}{2}\sqrt{2}\left(\sqrt{5}+1\right) + \sqrt{5-\sqrt{5}} - i\left(\frac{1}{2}\sqrt{2}\left(\sqrt{5}+1\right) - \sqrt{5-\sqrt{5}}\right)\right) + \\ &+ \frac{1}{8}\sqrt[3]{\frac{\sqrt{3}-i}{2}} \cdot \left(\frac{1}{2}\sqrt{2}\left(\sqrt{5}+1\right) + \sqrt{5-\sqrt{5}} + i\left(\frac{1}{2}\sqrt{2}\left(\sqrt{5}+1\right) - \sqrt{5-\sqrt{5}}\right)\right) \end{aligned}$$

For 2°, observe that $2° = 20° - 18°$, and $20° = \frac{\pi}{9}$. Hence:

$$\begin{aligned}\sin(2°) = \sin(20° - 18°) = \sin(20°)\cos(18°) - \cos(20°)\sin(18°) =\\ = \frac{1}{2i}(e^{i\frac{\pi}{9}} - e^{-i\frac{\pi}{9}})\cos(18°) - \frac{1}{2}(e^{i\frac{\pi}{9}} + e^{-i\frac{\pi}{9}})\sin(18°) =\\ = \frac{1}{2}e^{i\frac{\pi}{9}}(-\sin(18°) - i\cos(18°)) + \frac{1}{2}e^{-i\frac{\pi}{9}}(-\sin(18°) + i\cos(18°))\end{aligned}$$

and:

$$\begin{aligned}\cos(2°) = \cos(20° - 18°) = \cos(20°)\cos(18°) + \sin(20°)\sin(18°) =\\ = \frac{1}{2}(e^{i\frac{\pi}{9}} + e^{-i\frac{\pi}{9}})\cos(18°) + \frac{1}{2i}(e^{i\frac{\pi}{9}} - e^{-i\frac{\pi}{9}})\sin(18°) =\\ = \frac{1}{2}e^{i\frac{\pi}{9}}(\cos(18°) - i\sin(18°)) + \frac{1}{2}e^{-i\frac{\pi}{9}}(\cos(18°) + i\sin(18°))\end{aligned}$$

The term $e^{i\frac{\pi}{9}}$ is the principal cube root of $e^{i\frac{\pi}{3}} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, and the term $e^{-i\frac{\pi}{9}}$ is the principal cube root of $e^{-i\frac{\pi}{3}} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$, so they can be written as:

$$e^{i\frac{\pi}{9}} = \sqrt[3]{\frac{1 + i\sqrt{3}}{2}}, \qquad e^{-i\frac{\pi}{9}} = \sqrt[3]{\frac{1 - i\sqrt{3}}{2}}$$

The values of $\sin(18°)$ and $\cos(18°)$ are the following:

$$\begin{aligned}\sin(18°) &= \frac{1}{8}\left(\sqrt{5} - 1\right)\\ \cos(18°) &= \frac{1}{4}\sqrt{2}\sqrt{5 + \sqrt{5}}\end{aligned}$$

Altogether:

$$\sin(2°) = \frac{1}{8}\sqrt[3]{\frac{1 + i\sqrt{3}}{2}} \cdot \left(-\frac{\sqrt{5} - 1}{2} - i\sqrt{2}\sqrt{5 + \sqrt{5}}\right) + \frac{1}{8}\sqrt[3]{\frac{1 - i\sqrt{3}}{2}} \cdot \left(-\frac{\sqrt{5} - 1}{2} + i\sqrt{2}\sqrt{5 + \sqrt{5}}\right)$$

and:

$$\cos(2°) = \frac{1}{8}\sqrt[3]{\frac{1 + i\sqrt{3}}{2}} \cdot \left(\sqrt{2}\sqrt{5 + \sqrt{5}} - i\frac{\sqrt{5} - 1}{2}\right) + \frac{1}{8}\sqrt[3]{\frac{1 - i\sqrt{3}}{2}} \cdot \left(\sqrt{2}\sqrt{5 + \sqrt{5}} + i\frac{\sqrt{5} - 1}{2}\right)$$

Solution 2

The complex number $\cos(1°) + i\sin(1°)$ is:

$$\cos(1°) + i\sin(1°) = e^{i\left(\frac{\pi}{18} - \frac{\pi}{20}\right)} = e^{i\frac{\pi}{18}} \cdot \left(\cos(9°) - i\sin(9°)\right) =$$
$$= \frac{1}{4}\sqrt[3]{\frac{\sqrt{3}+i}{2}} \cdot \left(\frac{1}{2}\sqrt{2}\left(\sqrt{5}+1\right) + \sqrt{5-\sqrt{5}} \, - i\left(\frac{1}{2}\sqrt{2}\left(\sqrt{5}+1\right) - \sqrt{5-\sqrt{5}}\right)\right)$$

Then, $\sin(1°)$ and $\cos(1°)$ are its imaginary part and its real part, respectively.

The complex number $\cos(2°) + i\sin(2°)$ is:

$$\cos(2°) + i\sin(2°) = e^{i\left(\frac{\pi}{9} - \frac{\pi}{10}\right)} = e^{i\frac{\pi}{9}} \cdot \left(\cos(18°) - i\sin(18°)\right) =$$
$$= \frac{1}{4}\sqrt[3]{\frac{1+i\sqrt{3}}{2}} \cdot \left(\sqrt{2}\sqrt{5+\sqrt{5}} - i\,\frac{\sqrt{5}-1}{2}\right)$$

Then, $\sin(2°)$ and $\cos(2°)$ are its imaginary part and its real part, respectively.

Problem 1.3

Evaluate $\tan\left(\frac{1}{7}\pi\right) \cdot \tan\left(\frac{3}{7}\pi\right) \cdot \tan\left(\frac{5}{7}\pi\right)$.

Solution

To do

Problem 1.4

Determine the number of digits of 125^{100} in base 10.

Solution 1

The number of digits is 210. To prove this, it needs to be shown that $10^{209} \leq 125^{100} < 10^{210}$.

For the first inequality, first observe that $5^5 > 3{\cdot}2^{10}$ (since $5^5 = 3125$, and $3{\cdot}2^{10} = 3{\cdot}1024 = 3072$). Squaring yields: $5^{10} > 9 \cdot 2^{20} > 8 \cdot 2^{20}$. Raising to the power of 10 gives: $5^{100} > 8^{10} \cdot 2^{200} = 1024^3 \cdot 2^{200} > 10^9 \cdot 2^{200}$. Multiplying by 5^{200} yields: $125^{100} = 5^{300} = 5^{200} \cdot 5^{100} > 5^{200} \cdot 10^9 \cdot 2^{200} = 10^{209}$.

The second inequality is equivalent to $125^{10} < 10^{21}$, which follows by: $125^3 \cdot 125^7 < 128^3 \cdot 125^7 = 2^{21} \cdot 5^{21}$.

Solution 2

The number of digits is 210. To prove this, it needs to be shown that $10^{209} \leq 125^{100} < 10^{210}$.

For the first inequality, first observe that $5^5 > 3 \cdot 2^{10}$ (since $5^5 = 3125$, and $3 \cdot 2^{10} = 3 \cdot 1024 = 3072$). Squaring yields: $5^{10} > 9 \cdot 2^{20} > 8 \cdot 2^{20} = 2^{23}$. Multiplying by 5^{23} gives: $125^{11} = 5^{33} = 5^{10} \cdot 5^{23} > 2^{23} \cdot 5^{23} = 10^{23}$. Raising to the power of 9 gives $125^{99} > 10^{9 \cdot 23} = 10^{207}$. Finally, $125^{100} = 125 \cdot 125^{99} > 100 \cdot 125^{99} > 100 \cdot 10^{207} = 10^{209}$.

The second inequality is equivalent to $5^{10} < 10^7 = 2^7 \cdot 5^7$ (by raising to the power of 30), which is equivalent to $5^3 < 2^7$, which is true ($125 < 128$).

Solution 3

The number of digits of a positive integer n in base b is $\lfloor \log_b(n) + 1 \rfloor$. Thus, the required number of digits is $\lfloor 100 \log_{10}(125) + 1 \rfloor$, (and the problem is essentially that of evaluating $\log_{10}(125)$ to two decimal places).

First notice that $\log_{10}(2) > \frac{3}{10}$, since $2^{10} = 1024 > 1000 = 10^3$. Then, observe that $3 = \log_{10}(1000) = \log_{10}(2^3 \cdot 5^3) = \log_{10}(125) + 3\log_{10}(2)$. From this, the following upper bound is deduced: $100\log_{10}(125) = 300 - 300\log_{10}(2) < 300 - 300 \cdot \frac{3}{10} = 210$.

Now observe that $\log_{10}(125) = \log_{10}\left(100 \cdot \frac{5}{4}\right) = 2 + \log_{10}\left(\frac{5}{4}\right)$. The following steps show that $100\log_{10}\left(\frac{5}{4}\right) > 9$:

$$\left(\frac{5}{4}\right)^4 = \left(1 + \frac{1}{4}\right)^4 = \left(1 + \frac{1}{2} + \frac{1}{16}\right)^2 > 1 + 1 + \frac{1}{4} + \frac{1}{8} = 2 + \frac{3}{8}$$
$$\left(\frac{5}{4}\right)^8 > \left(2 + \frac{3}{8}\right)^2 > 4 + \frac{3}{2} = 5 + \frac{1}{2}$$
$$\left(\frac{5}{4}\right)^{16} > \left(5 + \frac{1}{2}\right)^2 > 25 + 5 = 30$$
$$\left(\frac{5}{4}\right)^{33} > \frac{5}{4} \cdot 30^2 > 1000$$
$$\left(\frac{5}{4}\right)^{100} > \left(\frac{5}{4}\right)^{99} > 1000^3 = 10^9$$

In conclusion: $100\log_{10}(125) = 200 + 100\log_{10}\left(\frac{5}{4}\right) > 200 + 9 = 209$.
Therefore, $209 < \log_{10}(125^{100}) < 210$. It follows that $\lfloor \log_{10}(125^{100}) + 1 \rfloor = 210$.

Comparisons

Problem 2.1

Determine which among $\log_2(3)$ and $\log_3(5)$ is largest.

Solution

Observe that:

$$2\log_2(3) = \log_2(9) > \log_2(8) = 3$$
$$2\log_3(5) = \log_3(25) < \log_3(27) = 3$$

Thus $\log_3(5) < \log_2(3)$.

Problem 2.2

For $N \in \mathbb{N}$, determine which among $\prod_{n=2}^{N} \log_3(2n)$ and $2\prod_{n=2}^{N} \log_3(2n-1)$ is largest. (What about with a different base?)

Solution

To do

Problem 2.3

Determine which among $\frac{8}{27}\pi$ and $\sin\left(\frac{8}{7}\right)$ is largest.

Solution

To do

Problem 2.4

Determine which among $\sqrt[3]{413}$ and $6 + \sqrt[3]{3}$ is largest.

Solution 1

Claim: $\sqrt[3]{413} > 7 + \frac{4}{9}$. This can be shown as follows:

$$\begin{aligned}\left(7+\frac{4}{9}\right)^3 &= 7^3 + 3\cdot 7^2 \cdot \frac{4}{9} + 3\cdot 7\cdot \frac{4^2}{9^2} + \frac{4^3}{9^3} = 343 + \frac{196}{3} + \frac{112}{27} + \frac{64/27}{27} = \\ &= 343 + 65 + \frac{1}{3} + 4 + \frac{4}{27} + \frac{2+10/27}{27} < 412 + \frac{1}{3} + \frac{9}{27} = \\ &= 412 + \frac{2}{3} < 413\end{aligned}$$

Claim: $6 + \sqrt[3]{3} < 7 + \frac{4}{9}$; that is: $\sqrt[3]{3} < 1 + \frac{4}{9}$. This can be shown as follows:

$$\left(1+\frac{4}{9}\right)^3 = 1 + 3\cdot\frac{4}{9} + 3\cdot\frac{4^2}{9^2} + \frac{4^3}{9^3} = 1 + \frac{4}{3} + \frac{16}{27} + \frac{64}{729} = 2 + \frac{1}{3} + \frac{9+7}{27} + \frac{64/27}{27} =$$
$$= 2 + \frac{1}{3} + \frac{1}{3} + \frac{7}{27} + \frac{2+10/27}{27} = 2 + \frac{2}{3} + \frac{9}{27} + \frac{10}{729} > 3$$

So, in conclusion:

$$6 + \sqrt[3]{3} < 7 + \frac{4}{9} < \sqrt[3]{413}$$

Solution 2

Let $\alpha = \sqrt[3]{413} - \sqrt[3]{3}$. Then: $\alpha^3 = 413 - 3\cdot\sqrt[3]{3}\cdot\sqrt[3]{413}\cdot\left(\sqrt[3]{413} - \sqrt[3]{3}\right) - 3 = 410 - 3\sqrt[3]{1239}\,\alpha$.

Therefore, α is a root of the polynomial function $p(x) = x^3 + 3\sqrt[3]{1239}x - 410$. This function is strictly increasing, as a function $\mathbb{R} \to \mathbb{R}$, hence α is its only real root, and, moreover, for every $x \in \mathbb{R}$, $x < \alpha$ if and only $p(x) < 0$, and $x > \alpha$ if and only $p(x) > 0$.

Putting $x = 6$, it needs to be determined whether $216 + 18\sqrt[3]{1239} - 410$ is positive or negative, which simplifies to determining whether 97 is, respectively, less or greater than $9\sqrt[3]{1239}$. Now:

$$\left(\frac{97}{9}\right)^3 = \left(10 + \frac{7}{9}\right)^3 > 1000 + 3\cdot 100\cdot\frac{7}{9} + 3\cdot 10\cdot\frac{49}{81} =$$
$$= 1000 + \frac{700}{3} + \frac{490}{27} > 1000 + 233 + 10 >$$
$$> 1239$$

It follows that $97 > 9\sqrt[3]{1239}$, so $p(6) < 0$, which means that $6 < \alpha = \sqrt[3]{413} - \sqrt[3]{3}$. So, in conclusion:

$$6 + \sqrt[3]{3} < \sqrt[3]{413}$$

Solution 3

Let the polynomial $p(x,y,z) = (x-y)^2 + (y-z)^2 + (z-x)^2$, and recall the polynomial identity:

$$x^3 + y^3 + z^3 - 3xyz = (x+y+z)\cdot\frac{1}{2}p(x,y,z)$$

Note that the polynomial p evaluated at real values is always non-negative, since it is a sum of squares. Substituting $x = \sqrt[3]{413}$, $y = -6$, $z = -\sqrt[3]{3}$ in the above identity gives:

$$\frac{1}{2}\left(\sqrt[3]{413} - \left(6 + \sqrt[3]{3}\right)\right)\cdot p\left(\sqrt[3]{413}, -6, -\sqrt[3]{3}\right) = 413 - 216 - 3 - 3\cdot 6\cdot\sqrt[3]{3\cdot 413} =$$
$$= 194 - 18\sqrt[3]{1239} =$$
$$= 2\cdot\left(97 - 9\sqrt[3]{1239}\right)$$

Hence, if $97 > 9\sqrt[3]{1239}$, then $\sqrt[3]{413}$ is greater than $6 + \sqrt[3]{3}$, and if $97 < 9\sqrt[3]{1239}$, then $\sqrt[3]{413} < 6 + \sqrt[3]{3}$. Now:

$$\begin{aligned}\left(\frac{97}{9}\right)^3 &= \left(10 + \frac{7}{9}\right)^3 > 1000 + 3 \cdot 100 \cdot \frac{7}{9} + 3 \cdot 10 \cdot \frac{49}{81} = \\ &= 1000 + \frac{700}{3} + \frac{490}{27} > 1000 + 233 + 10 > \\ &> 1239\end{aligned}$$

It follows that $97 > 9\sqrt[3]{1239}$. So, in conclusion:

$$6 + \sqrt[3]{3} < \sqrt[3]{413}$$

Addendum The values of the two expressions are approximately:

$$\begin{aligned}\sqrt[3]{413} &\approx 7.447034238 \\ 6 + \sqrt[3]{3} &\approx 7.442249570\end{aligned}$$

Their difference is approximately 0.004784668.

Problem 2.5

Prove that:

$$\sqrt[3]{3 + \sqrt[3]{3}} + \sqrt[3]{3 - \sqrt[3]{3}} < 2\sqrt[3]{3}$$

Solution 1

Let $r \in (0, 1)$, and consider the function $f : \mathbb{R}^+ \to \mathbb{R}^+ : x \mapsto x^r$. Let $a \in \mathbb{R}^+$, and consider the function $g : \mathbb{R}^+ \to \mathbb{R}^+ : x \mapsto f(x + a) - f(x)$.

Claim: g is decreasing.
This can be proven by showing that $g' < 0$. Now: $g'(x) = f'(x + a) - f'(x)$, so it is sufficient for f' to be decreasing, which is the case if $f'' < 0$. This is the case: $f''(x) = r(r - 1)x^{r-2} < 0$, because $r > 0$, $r - 1 < 0$, $x > 0$.

Since g is decreasing, then, in particular, $g(x) < g(x - a)$, that is: $f(x + a) - f(x) < f(x) - f(x - a)$, which can be rewritten as $f(x + a) + f(x - a) < 2f(x)$.

With $r = \dfrac{1}{3}$, $a = \sqrt[3]{3}$, and $x = 3$, the claimed inequality follows.

Solution 2

Let $r \in (0,1)$, and consider the function $f : \mathbb{R}^+ \to \mathbb{R}^+ : x \mapsto x^r$.

f is strictly concave (that is, $-f$ is strictly convex), since for all $x \in \mathbb{R}^+$ $f''(x) = r(r-1)x^{r-2} < 0$, because $r > 0$, $r - 1 < 0$, and $x > 0$.
Hence, for any $x, y \in \mathbb{R}^+$ with $x \neq y$, and for any $t \in (0,1)$: $f(tx + (1-t)y) > tf(x) + (1-t)f(y)$.
Putting $r = \dfrac{1}{3}$, $x = 3 + \sqrt[3]{3}$, $y = 3 - \sqrt[3]{3}$, $t = \dfrac{1}{2}$, the claimed inequality is obtained.

Problem 2.6

Prove that:

$$\sqrt{3 + 32\sin^2(15°)} + \cos(22°) + \cos(70°) + \cos(88°) + 2\sqrt{2}\sin(15°) > \frac{3}{2}\left(\cos(11°) + \cos(35°) + \cos(44°)\right)^2$$

Solution

To do

Equations

Problem 3.1

For $a \in \mathbb{R}$, find all real numbers $x \geq -a$ that satisfy:

$$\sqrt{a + \sqrt{a + x}} = x$$

Solution

To do

Problem 3.2

Find all $x \in \mathbb{R}$ that satisfy:

$$2\sqrt[3]{2x - 1} = x^3 + 1$$

Solution

To do

Problem 3.3

Let $a \in \mathbb{R}^+$. Find all $x \in \mathbb{R}^+$ that satisfy:

$$x^{x^{\cdot^{\cdot^{\cdot^{x^a}}}}} = a$$

for a given height of the power tower.

Solution

To do

Problem 3.4

Find all $x \in \mathbb{R}$ (or $\mathbb{C}$?) that satisfy:

$$x^4 - 14x^3 + 66x^2 - 115x + 66 + \frac{1}{4} = 0$$

Solution

To do

Problem 3.5

Find all $x, y \in \mathbb{R}$ that satisfy:

$$\begin{cases} y \cdot (x+y)^2 = 9 \\ y \cdot (x^3 - y^3) = 7 \end{cases}$$

Solution

To do

Problem 3.6

For each $n \in \mathbb{N}$, determine the set M_n of pairs $(a, b) \in \mathbb{R}^2$ such that the equation $x^2 - a = |x - b|$ has exactly n solutions in $\mathbb{R}$.

Describe the plot of each set M_n in $\mathbb{R}^2$.

Solution

To do

Problem 3.7

Investigate the following equation, for $a, b \in \mathbb{R}$:

$$2^a + 2^{-a} = b\cos(\pi a)$$

Solution

To do

Problem 3.8

Find all $x \in \mathbb{R}$ that satisfy:

$$\sin^7(x) + \frac{1}{\sin^3(x)} = \cos^7(x) + \frac{1}{\cos^3(x)}$$

(Are the numbers 3 and 7 important or could they be any?)

Solution

To do

Problem 3.9

Find all $x \in \mathbb{R}$ that satisfy:

$$\sin^{\frac{11}{7}}(x) + \cos^{\frac{19}{11}}(x) = \sqrt{\frac{19}{7}}$$

(Are the numbers 11, 7, 19 important or could they be any?)

Solution

To do

Problem 3.10

For $a, r \in \mathbb{R}$ with $a > 1$ and $\frac{1}{2} \leq r \leq \frac{1}{2}a$, find all $x \in \mathbb{R}$ that satisfy:

$$\left(1 - \frac{1}{a}\cos^2(x)\right)^r = \sin(x)$$

and describe the solution set when $r > \frac{1}{2}a$.

Solution

Let $b = 2r$; thus $1 \leq b \leq a$. Square both sides of the equation:

$$\left(1 - \frac{1}{a}\cos^2(x)\right)^b = \sin^2(x)$$

Make the substitution $y = 1 - \frac{1}{a}\cos^2(x)$. The range of possibe values of y is $\left[1 - \frac{1}{a}, 1\right]$.
Using that $\sin^2(x) = 1 - \cos^2(x)$ and $\cos^2(x) = a - ay$, the substitution yields the equation

$$y^b - ay + a - 1 = 0$$

Clearly, 1 is a solution. For $b = 1$ it is clearly the only solution (since $a \neq 1$). For $b > 1$, let $q : \mathbb{R}^+ \to \mathbb{R}$ be the function defined by:

$$q(y) = \begin{cases} \dfrac{y^b - ay + a - 1}{y - 1} = \dfrac{1 - y^b}{1 - y} - a & \text{for } y \neq 1 \\ b - a & \text{for } y = 1 \end{cases}$$

which is well defined and continuous everywhere, and satisfies the relation $y^b - ay + a - 1 = (y-1)q(y)$ for every $y \in \mathbb{R}^+$.

Claim: q is strictly increasing.
To prove this, first observe that q is differentiable, with derivative:

$$q'(y) = \begin{cases} \dfrac{(b-1)y^b - by^{b-1} + 1}{(y-1)^2} & \text{for } y \neq 1 \\ \dfrac{1}{2}b(b-1) & \text{for } y = 1 \end{cases}$$

It sufficies to show that q' is positive. $q'(1) = \frac{1}{2}b(b-1) > 0$, so it remains to prove that $q'(y) > 0$ for $y \neq 1$. For this, it is enough to prove that the numerator $h(y) = (b-1)y^b - by^{b-1} + 1$ is positive for $y \neq 1$ (because the denominator is $(y-1)^2$, which is always positive for $y \neq 1$).
The derivative $h'(y) = b(b-1)(y-1)y^{b-2}$ is negative for $0 < y < 1$, positive for $y > 1$, and is zero at 1, so 1 is a point of global minimum for h; thus $h(y) > h(1) = 0$ for $y \neq 1$, as required.

In particular, q is strictly increasing in $\left[1 - \frac{1}{a}, 1\right]$, thus it admits at most one root in that range.

- For $1 < b < a$, q does not have any roots in the interval, because $q(1) = b - a < 0$, so $q < 0$ on the whole interval (since it is increasing).
- For $b = a$, the root is 1, since $q(1) = b - a = 0$.
- For $b > a$, there is a root β in the range, and it is neither $1 - \frac{1}{a}$ nor 1. This is because $q\left(1 - \frac{1}{a}\right) < 0$ and $q(1) = b - a > 0$:

$$q\left(1 - \frac{1}{a}\right) = \frac{1 - \left(1 - \frac{1}{a}\right)^b}{1 - 1 + \frac{1}{a}} - a = \frac{a^b - (a-1)^b}{a^{b-1}} - a = -\frac{(a-1)^b}{a^{b-1}} < 0$$

So, in conclusion, in the interval $\left[1 - \frac{1}{a}, 1\right]$:

- for $\frac{1}{2} \le r \le \frac{1}{2}a$, the equation $y^{2r} - ay + a - 1 = 0$ admits only the solution $y = 1$;
- for $r > \frac{1}{2}a$, the equation $y^{2r} - ay + a - 1 = 0$ admits the two solutions $y = \beta$ and $y = 1$.

The solution $y = 1$ corresponds to $\cos(x) = 0$, that is $x = \frac{\pi}{2} + k\pi$ for $k \in \mathbb{Z}$. Of those values of x, the ones for which k is even are indeed solutions to the original equation, while those for which k is odd are not, due to $\sin(x)$ being negative in that case. So this leaves the solutions $x = \frac{\pi}{2} + 2m\pi$ for $m \in \mathbb{Z}$.

The solution $y = \beta$ corresponds to $\cos(x) = \pm\sqrt{a(1-\beta)}$. Of the corresponding values of x, those for which $\sin(x)$ is positive are solutions to the original equation, while the others are not.

Problem 3.11

Find all $x \in \mathbb{R}$ (or $\mathbb{C}$?) that satisfy:

$$\cot(x) = \sin\left(x + \frac{\pi}{4}\right)$$

Solution

To do

Problem 3.12

Find all $x \in \mathbb{R}$ that satisfy:

$$\sin^3(x)\cos\left(\frac{x}{2}\right) + \frac{1}{2}\sin(x)\sin\left(\frac{x}{2}\right)\left(1 + 2\cos\left(\frac{x}{2}\right)\right) - 6\sin^2\left(\frac{x}{2}\right) - 1 = 0$$

Solution

To do

Problem 3.13

Find all $x \in \mathbb{R}^+$ that satisfy:

$$\frac{1}{16^x} = \log_{\frac{1}{16}}(x)$$

Solution

To do

Inequations

Problem 4.1

Find all $x \in [-1, 1]$ that satisfy:

$$x \cdot \left(8\sqrt{1-x} + \sqrt{1+x}\right) \leq 11\sqrt{1+x} - 16\sqrt{1-x}$$

Solution

Notice that $x = -1$ does not satisfy the inequality; thus the range of values of x can be restricted to $(-1, 1]$. The inequality is equivalent to the following, obtained by dividing by $\sqrt{1+x}$, which is always strictly positive for $x \in (-1, 1]$:

$$x \cdot \left(8\frac{\sqrt{1-x}}{\sqrt{1+x}} + 1\right) \leq 11 - 16\frac{\sqrt{1-x}}{\sqrt{1+x}}$$

Let $y = \frac{\sqrt{1-x}}{\sqrt{1+x}}$. Hence $x = \frac{1-y^2}{1+y^2}$. The inequality becomes:

$$\frac{1-y^2}{1+y^2} \cdot (8y+1) \leq 11 - 16y$$

and it is equivalent to the following, obtained by multiplying by $1 + y^2$ (which is always strictly positive):

$$(1-y^2)(8y+1) \leq (1+y^2)(11-16y)$$

which, expanded, is:

$$8y^3 - 12y^2 + 24y - 10 \leq 0$$

The polynomial $8y^3 - 12y^2 + 24y - 10$ has a root at $y = \frac{1}{2}$, which yields the factorisation $8y^3 - 12y^2 + 24y - 10 = 2(2y-1)(2y^2 - 2y + 5)$. Thus, the inequality can be written as:

$$2(2y-1)(2y^2 - 2y + 5) \leq 0$$

The quantity $2y^2 - 2y + 5$ is always strictly positive, so the inequality is equivalent to $2y - 1 \leq 0$, that is: $2\frac{\sqrt{1-x}}{\sqrt{1+x}} \leq 1$. Since $\frac{\sqrt{1-x}}{\sqrt{1+x}}$ is non-negative, the inequality is equivalent to $4\frac{1-x}{1+x} \leq 1$, obtained by squaring. Since $x > -1$, the inequality is equivalent to $4(1-x) \leq 1 + x$, obtained by multiplying by $1 + x$, which is positive. This simplifies to $5x \geq 3$, that is: $x \geq \frac{3}{5}$.

In conclusion, the solution set is $\left[\frac{3}{5}, 1\right]$.

Problem 4.2

Find all $a \in \mathbb{R}$ such that for any $x \in \mathbb{R}^+$ the following holds:

$$ax^2 + 2x > 3a - 1$$

Solution

To do

Problem 4.3

Find all $x \in \mathbb{R}$ that satisfy:

$$2^{\sin(x)} + 2^{\cos(x)} \geq 2^{1-\frac{1}{\sqrt{2}}}$$

Solution

To do

Problem 4.4

Find all $x \in \mathbb{R}$ that satisfy:

$$\frac{1}{\sin^2(x)} < \frac{1}{x^2} + 1 - \frac{4}{\pi^2}$$

And determine

$$\lim_{x \to 0} \frac{1}{\sin^2(x)} - \frac{1}{x^2}$$

Solution

To do

Problem 4.5

Determine the largest $a \in \mathbb{R}^+$ such that for all $x \in \left(0, \frac{\pi}{2}\right]$ the following holds:

$$\operatorname{sinc}^a(x) > \cos(x)$$

Solution

To do

Problem 4.6

Find all $(x, y) \in (-3, 3) \times \mathbb{R}$ that satisfy:

$$3^y \log_3(9 - x^2) \leq 1 + 3^{2y}$$

Solution

The inequality can be equivalently written as

$$\log_3(9 - x^2) \leq 3^y + 3^{-y}$$

It is satisfied by all $(x, y) \in (-3, 3) \times \mathbb{R}$. In fact, $\log_3(9 - x^2)$ is maximum when $9 - x^2$ is maximum, which is when $x = 0$; this corresponds to a global maximum of $\log_3(9) = 2$.

And $3^y + 3^{-y}$ is minimum for $y = 0$, (since $3^y + 3^{-y} = \left(3^{y/2} - 3^{-y/2}\right)^2 + 2$) which corresponsds to a global minimum of $3^0 + 3^0 = 2$.

Therefore, for every x and y, the inequality $\log_3(9 - x^2) \leq 2 \leq 3^y + 3^{-y}$ holds.

Algebra and Number Theory

Problem 5.1

Prove that for any $\alpha \in \mathbb{R}$, α is irrational if and only if the set $\{n + m\alpha \mid n, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}$.

Solution

To do

Problem 5.2

For each $a \in \mathbb{Z}$, let $\mathrm{P}(a)$ be the set of prime divisors of a. Characterise the set:

$$S = \left\{(a,b) \in \mathbb{N}^2 \;\middle|\; \mathrm{P}(a) = \mathrm{P}(b),\ \mathrm{P}(a+1) = \mathrm{P}(b+1)\right\}$$

What about the set $T = \left\{(a,b) \in \mathbb{Z}^2 \mid \mathrm{P}(a) = \mathrm{P}(b),\ \mathrm{P}(a+1) = \mathrm{P}(b+1)\right\}$?

Solution

Clearly, all pairs of the form (a,a) for $a \in \mathbb{N}$ are in S; and for any $(a,b) \in S$, also (b,a) is in S.

The pair $(75, 1215)$ is in S, because $75 = 3 \cdot 5^2$, $1215 = 3^5 \cdot 5$, $76 = 2^2 \cdot 19$, $1216 = 2^6 \cdot 19$.

The following is a family of elements of S, parametrised by $n \in \mathbb{Z}^+$:

$$a_n = 2^{n+1} - 2 = 2 \cdot (2^n - 1), \qquad b_n = 2^{n+1} a_n = 2^{2n+2} - 2^{n+2}$$

In fact, $\mathrm{P}(a) = \{2\} \cup \mathrm{P}(2^n - 1)$ and $\mathrm{P}(b) = \{2\} \cup \mathrm{P}(a) = \mathrm{P}(a)$; and $b_n + 1 = \left(2^{n+1} - 1\right)^2 = (a_n + 1)^2$, so that $\mathrm{P}(b+1) = \mathrm{P}(a+1)$.

NOTE: This does not solve the problem yet.

Problem 5.3

Prove that for every $n \in \mathbb{Z}^+$:

$$\prod_{\substack{1 \le p \le n \\ p \text{ prime}}} p \le 4^{n-1}$$

Solution

To do

Problem 5.4

Prove that $\sin(10°)$, $\cos(10°)$, and $\tan(10°)$ are irrational and algebraic, determine their algebraic degrees over $\mathbb{Q}$, and determine their minimal polynomials over $\mathbb{Q}$.

Solution

From the angle triplication formula for the sine function, it follows that $\frac{1}{2} = \sin(30°) = -4\sin^3(10°) + 3\sin(10°)$. Hence, $\sin(10°)$ is a root of the polynomial $8x^3 - 6x + 1$. By the rational root theorem, the only possible rational roots of that polynomial are $1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{8}$. By inspection, none of those is a root of the polynomial. (Alternatively, the polynomial is irreducible by Eisenstein's criterion applied to the coefficients in reverse order, with the prime number 2). Hence $\sin(10°)$ is irrational, with minimal polynomial $x^3 - \frac{3}{4}x + \frac{1}{8}$, and algebraic degree 3.

From the power reduction formulas for the cosine function, it follows that $\cos^2(10°) = \frac{1}{2}\cos(20°) + \frac{1}{2}$. From the angle triplication formula for the cosine function, it follows that $\frac{1}{2} = \cos(60°) = 4\cos^3(20°) - 3\cos(20°)$. Hence $\cos(20°)$ is a root of the polynomial $8x^3 - 6x - 1$. By the rational root theorem, the only possible rational roots of that polynomial are $1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{8}$. By inspection, none of those is a root of the polynomial. (Alternatively, the polynomial is irreducible by Eisenstein's criterion applied to the coefficients in reverse order, with the prime number 2). Hence $\cos(20°)$ is irrational, and so $\cos(10°)$ is also irrational (since its square is irrational).
[TODO: minimal polynomial and degree.]

From the angle triplication formula for the tangent function, it follows that $\frac{1}{\sqrt{3}} = \tan(30°) = \frac{\tan^3(10°) - 3\tan(10°)}{3\tan^2(10°) - 1}$. Hence, $\tan(10°)$ is a root of the polynomial $3(x^3 - 3x)^2 - (3x^2 - 1)^2 = 3x^6 - 27x^4 + 33x^2 - 1$. This polynomial is irreducible over $\mathbb{Q}[x]$, by Eisenstein's criterion applied to the coefficients in reverse order, with the prime number 3. In particular, it has no rational roots, hence $\tan(10°)$ is irrational, with minimal polynomial $x^6 - 9x^4 + 11x^2 - \frac{1}{3}$, and algebraic degree 6.

Problem 5.5

1. Do there exist rational numbers $a, b > 0$ such that a^b is rational?
2. Do there exist rational numbers $a, b > 0$ such that a^b is irrational?
3. Do there exist $a \in \mathbb{R}^+$ rational and $b \in \mathbb{R}^+$ irrational such that a^b is rational?
4. Do there exist $a \in \mathbb{R}^+$ rational and $b \in \mathbb{R}^+$ irrational such that a^b is irrational?
5. Do there exist $a \in \mathbb{R}^+$ irrational and $b \in \mathbb{R}^+$ rational such that a^b is rational?
6. Do there exist $a \in \mathbb{R}^+$ irrational and $b \in \mathbb{R}^+$ rational such that a^b is irrational?
7. Do there exist irrational numbers $a, b \in \mathbb{R}^+$ such that a^b is rational?
8. Do there exist irrational numbers $a, b \in \mathbb{R}^+$ such that a^b is irrational?

Solution 1

Yes to all.

1. For any rational a and integer b, a^b is rational.

2. An example is: $a = 5$, $b = \frac{1}{3}$, which give $a^b = \sqrt[3]{5}$.
3. An example is $a = 2$ and $b = \log_2(3)$, which give $a^b = 3$.
4. An example is $a = 2$ and $b = \frac{1}{2}\log_2(3)$, which give $a^b = \sqrt{3}$.
5. An example is $a = \sqrt[3]{4}$ and $b = \frac{3}{2}$, which give $a^b = 2$.
6. An example is $a = \sqrt{3}$ and $b = 3$, which give $a^b = 3\sqrt{3}$.
7. An exmaple is $a = \sqrt{2}$ and $b = 2\log_2(3)$, which give $a^b = 3$.
8. An example is $a = \sqrt{2}$ and $b = \log_2(3)$, which give $a^b = \sqrt{3}$.

Note: $\log_2(3)$ is irrational because no power of 2 is a power of 3 (with non-zero integer exponents).

Solution 2

Non explicit solutions for 5, 7 and 8 are the following:

7. If $\sqrt{2}^{\sqrt{2}}$ is rational, then $a = \sqrt{2}$ and $b = \sqrt{2}$ are irrational numbers such that a^b is rational; otherwise, if $\sqrt{2}^{\sqrt{2}}$ is irrational, then $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$ are irrational numbers such that $a^b = 2$ is rational.

5, 8. For any $a \in \mathbb{R}^+ \setminus \{1\}$, the function $r : \mathbb{R} \setminus \mathbb{Q} \to \mathbb{R} : x \mapsto a^x$ is a bijection on its image hence $\#(\mathrm{im}(r)) = \#(\mathbb{R} \setminus \mathbb{Q}) = \#(\mathbb{R}) > \#(\mathbb{Q})$; in particular $\mathrm{im}(r) \not\subseteq \mathbb{Q}$. This means that for every $a > 0$, $a \neq 1$, there exist a continuum of irrational numbers $b > 0$ such that $a^b \in \mathbb{R} \setminus \mathbb{Q}$.

Problem 5.6

The digit expansion of a number $a \in (0, 1)$ has 0 as first digit, then for every $n \in \mathbb{N}$, the digits $(2^n + 1)$-th to 2^{n+1}-th are the opposite of the digits 1-st to 2^n-th, respectively, where the opposite of the digit 1 is the digit 0, and viceversa. Prove that a is irrational.

Solution

To do

Problem 5.7

For which integers $n \geq 1$ does there exist a regular n-gon in $\mathbb{R}^2$ whose vertices are rational points? (That is, whose vertices are in $\mathbb{Q}^2$).

(What about in $\mathbb{R}^m$?) (What about regular solids in higher dimensions?)

Solution

Only for $n = 1, 2, 4$.

To prove this, observe that the area of a polygon with rational vertices is rational (this follows from Pick's theorem or the "shoelace" area formula).

The area of a regular n-gon of side length l is $\frac{1}{4}nl^2\cot(\frac{\pi}{n})$. Observe that l^2 is rational if the vertices have rational coordinates (by the Pythagorean theorem).

To prove that for a given $n \in \mathbb{N}$, a regular n-gon cannot have rational vertices, it sufficies to show that $\cot(\frac{\pi}{n})$ is irrational, or, equivalently, that $\tan(\frac{\pi}{n})$ is irrational (the result will then follow by contradiction). Observe that this is the case for $n = 8$: $\tan(\frac{\pi}{8}) = \sqrt{2} - 1$ is irrational. Now it is sufficient to prove the result for $n \geq 3$ prime, since the existence of a regular n-gon with rational vertices implies the existence of a regular d-gon with rational vertices for each $d \geq 3$ divisor of n. (Numbers that do not have odd prime factors are powers of 2, so they are taken care of by the case $n = 8$, while $n = 1, 2, 4$ are the only exception).

For each $n \in \mathbb{N}$, define the polynomials p_n and q_n as follows:

$$p_n(x) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{2k+1} x^{2k+1}$$
$$q_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{2k} x^{2k}$$

Observe that these polynomials satisfy the following relations, for every $n \in \mathbb{N}$:

$$p_{n+1}(x) = p_n(x) + x \cdot q_n(x)$$
$$q_{n+1}(x) = q_n(x) - x \cdot p_n(x)$$

In fact:

$$\begin{aligned} p_n(x) + x \cdot q_n(x) &= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{2k+1} x^{2k+1} + \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{2k} x^{2k+1} = \\ &= \sum_{k=0}^{+\infty} (-1)^k \left(\binom{n}{2k} + \binom{n}{2k+1} \right) x^{2k+1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n+1}{2k+1} x^{2k+1} = \\ &= p_{n+1}(x) \end{aligned}$$

and:

$$
\begin{aligned}
q_n(x) - x \cdot p_n(x) &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{2k} x^{2k} - \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{2k+1} x^{2k+2} = \\
&= \sum_{k \in \mathbb{Z}} (-1)^k \left(\binom{n}{2k-1} + \binom{n}{2k} \right) x^{2k} = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^k \binom{n+1}{2k} x^{2k} = \\
&= q_{n+1}(x)
\end{aligned}
$$

Now observe that for each $n \in \mathbb{N}$, the polynomial p_n is a multiple of the polynomial x: in fact $p_n(x) = x \cdot s_n(x)$, where:

$$
s_n(x) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{2k+1} x^{2k}
$$

The constant term of s_n is $\binom{n}{1} = n$. Additionally, for odd n, the leading coefficient of s_n is $\pm\binom{n}{n} = \pm 1$. This implies that when r is an odd prime, the only possible rational roots of s_r are $1, -1, r, -r$ (by the rational root theorem). It is immediate to verify that none of those is a root of the polynomial: 1 and -1 are ruled out by reducing modulo r (using the fact that the binomial coefficients $\binom{r}{a}$ are multiples of r for $1 \leq a \leq r-1$, due to r being prime):

$$
\begin{aligned}
s_r(1) &= \pm 1 + \sum_{k=0}^{\frac{r-3}{2}} (-1)^k \binom{r}{2k+1} \cong \pm 1 \ncong 0 \pmod{r} \\
s_r(-1) &= \pm 1 + \sum_{k=0}^{\frac{r-3}{2}} (-1)^k \binom{r}{2k+1} \cong \pm 1 \ncong 0 \pmod{r}
\end{aligned}
$$

r and $-r$ are ruled out by reducing modulo r^2:

$$
\begin{aligned}
s_r(r) &= r + \sum_{k=1}^{\frac{r-1}{2}} (-1)^k \binom{r}{2k+1} r^{2k} \cong r \ncong 0 \pmod{r^2} \\
s_r(-r) &= r + \sum_{k=1}^{\frac{r-1}{2}} (-1)^k \binom{r}{2k+1} r^{2k} \cong r \ncong 0 \pmod{r^2}
\end{aligned}
$$

In conclusion, for r odd prime, the only rational root of the polynomial p_r is 0 (with multiplicity 1).

Now observe that for every $\theta \in \mathbb{R}$ and for every $n \in \mathbb{N}$ such that $\theta, n\theta \in \mathrm{dom}(\tan)$:

$$
\tan(n\theta) = \frac{p_n(\tan(\theta))}{q_n(\tan(\theta))}
$$

This fact can be seen by induction: it is trivial for $n = 0$, since $p_0 = 0$ and $q_0 = 1$, and $\tan(0) = 0$; then, inductively:

$$\tan((n+1)\theta) = \tan(n\theta + \theta) = \frac{\tan(n\theta) + \tan(\theta)}{1 - \tan(n\theta) \cdot \tan(\theta)} = \frac{\frac{p_n(\tan(\theta))}{q_n(\tan(\theta))} + \tan(\theta)}{1 - \frac{p_n(\tan(\theta))}{q_n(\tan(\theta))} \cdot \tan(\theta)} =$$
$$= \frac{p_n(\tan(\theta)) + \tan(\theta) \cdot q_n(\tan(\theta))}{q_n(\tan(\theta)) - \tan(\theta) \cdot p_n(\tan(\theta))} = \frac{p_{n+1}(\tan(\theta))}{q_{n+1}(\tan(\theta))}$$

Now observe that for $n \geq 1$, $\tan(\frac{\pi}{n})$ is a root of the polynomial p_n, since:

$$0 = \tan(\pi) = \tan\left(n \cdot \frac{\pi}{n}\right) = \frac{p_n(\tan(\frac{\pi}{n}))}{q_n(\tan(\frac{\pi}{n}))}$$

Moreover, since $\tan(\frac{\pi}{n}) \neq 0$ for $n \geq 2$, it follows that for r odd prime, $\tan(\frac{\pi}{r})$ is not a rational root of p_r. Thus it is irrational.

Problem 5.8

A square of side length 1 is given on the plane. Does there exist a point on the plane whose distances to the vertices of the square are all rational?
(What about other polygons?) (What about regular solids in any dimension?) (What is the maximum number of distances that can be rational?)

Solution

To do

Problem 5.9

What is the largest cardinal α such that there exists a set $S \subseteq \mathbb{R}^2$ with $\#(S) = \alpha$, no three elements of which are collinear, and such that for every $p, q \in S$ the distance $\mathrm{d}(p, q)$ is integer?
(What about in higher dimension?)

Solution

To do

Problem 5.10

Determine all pairs of positive rational numbers (a, b) such that $a^b = b^a$.

Solution

To do

Problem 5.11

Determine all the pairs $(a, b) \in \mathbb{N}^2$ such that $a^2 + (a+1)^2 = b^2$.

Solution

To do

Problem 5.12

Determine all $x, y \in \mathbb{Q}\left[\sqrt{2}\right]$ such that

$$x^2 + y^2 = 5 + 4\sqrt{2}$$

Solution

To do

Problem 5.13

Rationalise the denominator in the following fraction:

$$\frac{1}{\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}}$$

Equivalently:
find two non-zero polynomials $s(x, y, z)$, $t(x, y, z)$ such that $t(x^3, y^3, z^3) = (x + y + z) \cdot s(x, y, z)$.

Solution

Consider the polynomial identity:

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

Let the polynomial $p(x, y, z) = x^2 + y^2 + z^2 - xy - yz - zx$. Then:

$$\frac{1}{\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}} = \frac{1}{\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}} \cdot \frac{p\left(\sqrt[3]{a}, \sqrt[3]{b}, \sqrt[3]{c}\right)}{p\left(\sqrt[3]{a}, \sqrt[3]{b}, \sqrt[3]{c}\right)} = \frac{p\left(\sqrt[3]{a}, \sqrt[3]{b}, \sqrt[3]{c}\right)}{a + b + c - 3\sqrt[3]{abc}}$$

The new denominator has only one radical. Let $u = a + b + c$ and $v = 3\sqrt[3]{abc}$. Now:

$$\frac{1}{\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}} = \frac{p\left(\sqrt[3]{a},\, \sqrt[3]{b},\, \sqrt[3]{c}\right)}{u - v} \cdot \frac{u^2 + uv + v^2}{u^2 + uv + v^2} = \frac{p\left(\sqrt[3]{a},\, \sqrt[3]{b},\, \sqrt[3]{c}\right) \cdot (u^2 + uv + v^2)}{u^3 - v^3}$$

Now the denominator is free of radicals; in conclusion, the fraction can be written as:

$$\frac{\left(\sqrt[3]{a^2} + \sqrt[3]{b^2} + \sqrt[3]{c^2} - \sqrt[3]{ab} - \sqrt[3]{bc} - \sqrt[3]{ca}\right) \cdot \left((a+b+c)^2 + 3(a+b+c)\sqrt[3]{abc} + 9\sqrt[3]{a^2b^2c^2}\right)}{(a+b+c)^3 - 27abc}$$

Observe that the numerator can be further factorised: let the polynomials $q(x, y) = x^2 - xy + y^2$ and $s(x, y, z) = p(x, y, z) \cdot q(x + y, z) \cdot q(y + z, x) \cdot q(z + x, y)$. Then, the numerator is $s\left(\sqrt[3]{a},\, \sqrt[3]{b},\, \sqrt[3]{c}\right)$. And, letting the polynomial $t(x, y, z) = (x + y + z)^3 - 27xyz$, it holds that $t(x^3, y^3, z^3) = (x + y + z) \cdot s(x, y, z)$.

Analysis

Problem 6.1

Let $f : \mathbb{R} \to \mathbb{R}$ be a monotonically increasing function. Let $a, b \in \mathbb{R}$ with $a < b$. Determine which points $c \in (a, b)$ minimize the value:

$$\int_a^c f(x) - f(a)\, \mathrm{d}x + \int_c^b f(b) - f(x)\, \mathrm{d}x$$

Solution

To do

Problem 6.2

Let $f : [0, 1] \to \mathbb{R}^+$ be a continuous function. Let $a, b \in \mathbb{R}^+$ such that $a \le f \le b$. Prove that:

$$ab \int_0^1 \frac{1}{f(x)}\, \mathrm{d}x \;\le\; a + b - \int_0^1 f(x)\, \mathrm{d}x$$

Solution

To do

Problem 6.3

Let $a > 1$ be a real number. Let $I \subseteq \mathbb{R}$ be an interval. Find all functions $f : I \to \mathbb{R}$ such that for any $x, y \in \mathbb{R}$ the following holds:

$$f(x) - f(y) \leq |x - y|^a$$

Solution 1

Let f be a function satisfying the stated condition. Let $x, y \in I$. Then $f(x) - f(y) \leq |x - y|^a$, and also $f(y) - f(x) \leq |y - x|^a = |x - y|^a$. Hence, for any $x, y \in I$, the relation $|f(x) - f(y)| \leq |x - y|^a$ holds. It follows that f is differentiable and its derivative is 0 everywhere, since for any $x \in I$ (using that $a - 1 > 0$):

$$\lim_{y \to x} \left| \frac{f(x) - f(y)}{x - y} \right| \leq \lim_{y \to x} |x - y|^{a-1} = 0$$

Therefore, every function f that satisfies the given condition is constant. Viceversa, every constant function clearly satisfies the condition.

Solution 2

Let f be a function satisfying the stated condition. Observe that for any $x, y \in \mathbb{R}$ it holds that $|f(x) - f(y)| \leq |x - y|^a$, since $f(x) - f(y) \leq |x - y|^a$ and $f(y) - f(x) \leq |y - x|^a = |x - y|^a$.

Let $x, y \in I$. Let $\varepsilon > 0$. Let $n \in \mathbb{N}$ such that $n^{a-1} > \frac{|y-x|^a}{\varepsilon}$. Let $x_0, \dots, x_n$ be a uniform partition of the interval between x and y: $x_0 = x$, and for each $k \in \{1, \dots, n\}$ $x_k = x_{k-1} + \frac{y-x}{n}$. In particular, $x_n = y$. Then:

$$|f(y) - f(x)| \leq \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \leq \sum_{k=1}^{n} |x_k - x_{k-1}|^a = n \cdot \left| \frac{y - x}{n} \right|^a = \frac{|y - x|^a}{n^{a-1}} < \varepsilon$$

So, for any $x, y \in I$ and for any $\varepsilon > 0$ it holds that $|f(y) - f(x)| < \varepsilon$. Hence for any $x, y \in I$, $f(y) = f(x)$. This means that f is constant. Therefore, every function f that satisfies the given condition is constant. Viceversa, every constant function clearly satisfies the condition.

Problem 6.4

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for any $x \in \mathbb{R}$ the following holds:

$$f(f(x)) = x^2 - 2$$

Solution

To do

Problem 6.5

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. If $\lim\limits_{n\to+\infty}(a_{n+1}-a_n)=0$, then does $\lim\limits_{n\to+\infty} a_n$ exist (finite or infinite)?

Solution

To do

Problem 6.6

Determine whether the following series converges. If it does, determine its value and its rate of convergence.

$$\sum_{n=1}^{+\infty}\frac{1}{n^3+3n^2+2n}$$

Solution

For each $N\in\mathbb{N}$, using partial fraction decomposition:

$$\begin{aligned}\sum_{n=1}^{N}\frac{1}{n^3+3n^2+2n} &= \sum_{n=1}^{N}\frac{1}{n(n+1)(n+2)} = \frac{1}{2}\sum_{n=1}^{N}\frac{1}{n}-\frac{2}{n+1}+\frac{1}{n+2} = \\ &= \frac{1}{2}\sum_{n=1}^{N}\frac{1}{n}+\frac{1}{2}\sum_{n=3}^{N+2}\frac{1}{n}-\sum_{n=2}^{N+1}\frac{1}{n} = \\ &= \frac{1}{2}+\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{N+1}+\frac{1}{2}\cdot\frac{1}{N+2}-\frac{1}{2}-\frac{1}{N+1}+\sum_{n=3}^{N}\frac{1}{n}-\frac{1}{n} = \\ &= \frac{1}{4}-\frac{1}{2}\cdot\frac{1}{N+1}+\frac{1}{2}\cdot\frac{1}{N+2} \\ &= \frac{1}{4}-\frac{1}{2(N+1)(N+2)}\end{aligned}$$

Thus the series converges to $\frac{1}{4}$, and

$$\left|\frac{1}{4}-\sum_{n=1}^{N}\frac{1}{n^3+3n^2+2n}\right| = \frac{1}{2(N+1)(N+2)} \sim \frac{1}{2N^2}$$

Problem 6.7

For $a,b,c,d\in\mathbb{R}$, what is the minimum value of $(a-d)^2+(b-c)^2$ under the constraints $a^2+4b^2=4$ and $cd=4$? And when is the minimum achieved?

Solution

To do

Plane Geometry

Problem 7.1

Let ABC be a triangle, with $\widehat{ABC} = 80°$. Let O be a point inside ABC such that $\widehat{OAC} = 10°$ and $\widehat{OCA} = 30°$.
Express the angle $\widehat{ABO}$ in terms of $\frac{\overline{OB}}{\overline{AC}}$.

Solution

To do

Problem 7.2

What is the maximum area for a triangle whose angle bisectors are all less than or equal to 1 in length? And when is the maximum achieved?
(What about the minimum area of a triangle whose bisectors are longer than or equal to 1?)

Solution

To do

Problem 7.3

Four circles on a plane are mutually tangent to each other. The points of tangency are all distinct. Three of the circles have collinear centers. Determine the distance between the center of fourth circle and the line through the centers of the others, in terms of the radius of the fourth circle.
(There are two cases: one for internal tangency and one for external tangency)
(What about for spheres in higher dimension?)

Solution

To do

Problem 7.4

Let ABC be a triangle. Let γ be its circumcircle. Let $\alpha_1, \alpha_2, \alpha_3$ be circles such that α_1 is tangent to $\overline{BC}$, to $\overline{CA}$, and to γ; α_2 is tangent to $\overline{AB}$, to $\overline{CA}$, and to γ; α_3 is tangent to $\overline{AB}$, to $\overline{BC}$, and to γ. Determine the radius of γ, given the radii of $\alpha_1, \alpha_2, \alpha_3$.

Distinguish all combinations of internal and external tangency between the circles.

Alernative formulation.

Three circles are each tangent to a (distinct) unordered pair of (distinct) sides of a triangle and to the circumcircle of the triangle. Determine the radius of the circumcircle, given the radii of the three circles.

Distinguish all combinations of internal and external tangency between the circles.

Solution

To do

Problem 7.5

Let ABC be an equilateral triangle, and let O be a point inside it. Show that the lengths $\overline{AO}$, $\overline{BO}$, $\overline{CO}$ can be the side lengths of a triangle, and determine the measures of the internal angles in such triangle, in terms of $\widehat{AOB}$, $\widehat{BOC}$ and $\widehat{COA}$.
(What about in arbitrary dimensional simplex?)

Solution

To do

Problem 7.6

Prove that a quadrilateral $ABCD$ is a rhombus if and only if the triangles AOB, BOC, COD, DOA are isoperimetetric, where O is the intersection of the diagonal lines AC and BD.

Solution

To do

Problem 7.7

Given a triangle, let R be the radius of its circumscribed circle, and r the radius of its inscribed circle. Determine the distance s between the centers of the two circles.

Determine the set of possible values of s among all triangles that have a fixed circumradius R; when are the extremes achieved?
(What about in arbitrary-dimensional simplex?)

Solution

To do

Problem 7.8

Given two intersecting lines r, s on a plane, and a real number $a \geq 0$, find the locus of points P of the plane such that $\mathrm{d}(P, r) + \mathrm{d}(P, s) = a$.
(What about in arbitrary dimension?) (What about intersection of three planes? What about $n+1$ n-hyperplanes?)

Solution

To do

Problem 7.9

Prove that the area of a quadrilateral with side lengths a, b, c, d which admits both inscribed and circumscribed circles is $\sqrt{abcd}$.
(What about the viceversa?)

Solution

To do

Problem 7.10

Determine the shortest networks that connect the four vertices of a square to each other.
(What about other configurations of points?)

Solution

To do

Problem 7.11

For any partition U of $\mathbb{R}^2$, let $D_U = \{a \in \mathbb{R}^+ \mid \exists\, T \in U \text{ such that } \exists\, p, q \in T \text{ with } \mathrm{d}(p,q) = a\}$. What is the largest cardinal α such that for every partition U of $\mathbb{R}^2$ with $\#(U) = \alpha$, the set D_U is the whole $\mathbb{R}^+$?

(What about partitions of $\mathbb{R}^m$?)

Solution

To do

Problem 7.12

Prove that a quadrilateral is cyclic if and only if the perpendiculars to each side passing through the midpoint of the opposite side are concurrent.

Solution

To do

Problem 7.13

Determine the quadrilateral with the largest area, given the lengths of its four sides, in order.

Solution

To do

Problem 7.14

Let $\overline{AB}$ be a chord in a circle, and let M be its midpoint. Let $\overline{CD}$ and $\overline{EF}$ be two other chords in the circle that pass through the point M, with C and F on opposite sides of $\overline{AB}$. Prove that $\overline{CF}$ intersects $\overline{AB}$ at a point P, and $\overline{ED}$ intersects $\overline{AB}$ at a point Q, on opposite sides of M, such that $\overline{MP} \cong \overline{MQ}$.

And prove that one and only one of the following holds:

- $\overline{CE}$ and $\overline{DF}$ are parallel to $\overline{AB}$;
- the line CE intersects the line AB at a point P', and the line DF intersects the line AB at a point Q', which are on opposite sides of M, such that $\overline{MP'} \cong \overline{MQ'}$.

(TODO: state the problem in projective geometry.)

Solution

To do

Problem 7.15

Given a triangle, determine a line that halves both its area and its perimeter.
(How many such lines are there in a given triangle?) (Can the construction be carried out with straightedge and compass?)

Solution

To do

Problem 7.16

Let $ABCD$ be a trapezoid with bases $\overline{AB}$ and $\overline{CD}$. Given a point $P \in \overline{AB}$, determine two points $Q_1, Q_2 \in \overline{CD}$ that, respectively, maximize the area of the quadrilateral intersection of the triangles ABQ_1 and CDP, and minimize the area of the quadrilateral intersection of the triangles ABQ_2 and CDP.

Solution

To do

Problem 7.17

Let a, b, c be the side lengths of a triangle, and let α, β, γ be the measures of their opposite angles, respectively. Prove that:

$$\frac{b+c-2a}{\sin\left(\frac{\alpha}{2}\right)} + \frac{c+a-2b}{\sin\left(\frac{\beta}{2}\right)} + \frac{a+b-2c}{\sin\left(\frac{\gamma}{2}\right)} \geq 0$$

Solution

To do

Problem 7.18

Let α, β, γ be the measures of the internal angles in a triangle. What is the maximum value of the quantity $\sqrt{\sin(\alpha)} + \sqrt{\sin(\beta)} + \sqrt{\sin(\gamma)}$, and when is it achieved?

Solution

To do

Problem 7.19

Let a, b, c be the side lengths of a triangle, and let α, β, γ be the measures of their opposite angles, respectively. What are the minimum and maximum values of $\frac{a\cdot\alpha+b\cdot\beta+c\cdot\gamma}{a+b+c}$, and when are they achieved?

Solution

To do

Problem 7.20

For a point P inside an equilateral (or regular?) hexagon of side length 1, what is the maximum sum of the distances between P and the vertices of the hexagon, and when is it achieved?
(What about in other polygons?)

Solution

To do

Problem 7.21

A circle is given on a plane. Given two points on the plane, construct a circle that passes through those two points and is tangent to the first circle.
(Is the problem asking for a straightedge and compass construction?)
(What about for spheres in arbitrary dimension, and an appropriate number of points?)

Solution

Consider a coordinate system with the two given points lying on the horizontal axis at coordinates $A : (-a, 0)$ and $B : (a, 0)$, for some $a \neq 0$. Let $C : (b, c)$ be the center of the given circle, and let $s > 0$ be the radius.
The wanted circle passes through A and B, so its center must lie on the perpendicular bisector of the two points, which is the vertical axis; so it has coordinates $P : (0, d)$, for some $d \in \mathbb{R}$. Let r be its radius. The tangency condition between the two circles is equivalent to $\mathrm{d}(P, C) = r + s$ (for external tangency) or $\mathrm{d}(P, C) = r - s$ (when the first circle is internally tangent to the wanted circle) or $\mathrm{d}(P, C) = s - r$ (when the wanted circle is internally tangent to the first circle). All in all, the tangency conditions can be

summarised as $\mathrm{d}(P,C)^2 = (r \pm s)^2$. Together with the condition $r = \mathrm{d}(P,A) = \mathrm{d}(P,B)$, this yields the following system of equations, in the unknowns d and r:

$$\begin{cases} r^2 = a^2 + d^2 \\ r^2 + s^2 \pm 2rs = b^2 + (c-d)^2 \end{cases}$$

Substituting $a^2 + d^2$ for r^2 in the second equation, and canceling the term d^2 yields:

$$a^2 + s^2 \pm 2rs = b^2 + c^2 - 2cd$$

So:

$$\mp r = \frac{c}{s}d + \frac{1}{2s}(a^2 - b^2 - c^2 + s^2)$$

Substituting in the first equation yields a quadratic equation in d:

$$(c^2 - s^2)d^2 + c(a^2 - b^2 - c^2 + s^2)d + \frac{1}{4}(a^2 - b^2 - c^2 + s^2)^2 - a^2 s^2 = 0$$

If $c^2 = s^2$, the equation is actually linear:

$$c(a^2 - b^2)d + \frac{1}{4}(a^2 - b^2)^2 - a^2 s^2 = 0$$

Note that $a^2 - b^2$ cannot be 0, since it would imply $a = 0$ or $s = 0$. Also, $c \neq 0$, since $c^2 = s^2 \neq 0$. The solution in this case is then:

$$d = \frac{a^2 s^2 - \frac{1}{4}(a^2 - b^2)^2}{c(a^2 - b^2)} = \frac{a^2 c}{a^2 - b^2} - \frac{a^2 - b^2}{4c}$$

Observe that $c^2 = s^2$ if and only if the line AB is tangent to the first circle. In this case, the line AB can be considered a degenerate second solution circle.

In the case $c^2 \neq s^2$, the solutions are:

$$d = \frac{-c\left(a^2-b^2-c^2+s^2\right) \pm \sqrt{c^2(a^2-b^2-c^2+s^2)^2-(c^2-s^2)(a^2-b^2-c^2+s^2)^2+4(c^2-s^2)a^2s^2}}{2(c^2-s^2)}$$

$$= \frac{-c\left(a^2-b^2-c^2+s^2\right) \pm s\sqrt{(a^2-b^2-c^2+s^2)^2+4a^2c^2-4a^2s^2}}{2(c^2-s^2)} =$$

$$= \frac{-c\left(a^2-b^2-c^2+s^2\right) \pm s\sqrt{(a^2+b^2+c^2-s^2)^2-4a^2b^2}}{2(c^2-s^2)} =$$

$$= \frac{-c\left(a^2-b^2-c^2+s^2\right) \pm s\sqrt{\left((a+b)^2+c^2-s^2\right)\left((a-b)^2+c^2-s^2\right)}}{2(c^2-s^2)} =$$

$$= \frac{-c\left(a^2-b^2-c^2+s^2\right) \pm s\sqrt{\left(\mathrm{d}(A,C)^2-s^2\right)\left(\mathrm{d}(B,C)^2-s^2\right)}}{2(c^2-s^2)}$$

Observe that $\left((a+b)^2+c^2-s^2\right)\left((a-b)^2+c^2-s^2\right)$ is positive if and only if A and B are both inside the first circle or both outside. It is 0 if and only if at least one among the points A and B lies on the circle.

In the general case, there are two circles that satisfy the stated conditions, and they are each uniquely determined by the coordinate d.

If the problem asks to perform the construction with only straightedge and compass, then it can be achieved by solving the quadratic equation for d geometrically, via, for example, the Carlyle circle construction, since all coefficients are constructible. Alternatively, the construction can be carried out by following the formula for d, since it only involves operations that can be performed with straightedge and compass. Note that it can be assumed that, for example, $s = 1$, since everything is scale-invariant.

In the case $s^2 = c^2$, the construction can also be performed with straightedge and compass, since the formula for d involves constructible operations only.

Problem 7.22

Prove that if a triangle and a square are circumscribed about the same circle, then the portion of the square contained inside the triangle makes up more than half of the perimeter of the square.
(Note that the triangle is generic: it is not necessarily equilateral).

Solution

To do

Problem 7.23

Let ABC be a triangle. Let M be the midpoint of $\overline{AC}$. Let $\overline{CL}$ be the angle bisector of $\widehat{BCA}$, with $L \in \overline{AB}$. Let P be the intersection point of $\overline{CL}$ and $\overline{BM}$. Prove that $\frac{\overline{CP}}{\overline{PL}} - \frac{\overline{AC}}{\overline{CB}} = 1$.

Solution

To do

Problem 7.24

How many unordered pairs of triangles have as union a given quadrilateral?

Solution

To do

Problem 7.25

n segments are given on the plane. Prove that the number of triangles whose sides are among those segments is $O(n^{3/2})$.

Solution

To do

Problem 7.26

Determine the largest $a \in \mathbb{R}^+$ such that every closed convex subset of $\mathbb{R}^2$ of area 1 contains a triangle of area a.

Solution

To do

Problem 7.27

Let $ABCDE$ be a convex pentagon. The triangles ABC, BCD, CDE, DEA, EAB all have area 1. Determine the area of the pentagon.

Solution

To do

Solid Geometry

Problem 8.1

Prove that if a sphere is tangent to all the edges of a three-dimensional quadrilateral, then the points of tangency are coplanar.
(Check if and how this generalises to higher dimensions.)

Solution

To do

Problem 8.2

Show that if in a tetrahedron the sums of lengths of opposite edges are all equal, then the sums of opposite dihedral angles are all equal.

Solution

To do

Problem 8.3

Find an equivalent condition for the bisectors of two trihedral angles of a tetrahedron to intersect.
(Note: a bisector of a trihedral angle is the locus of points that are equidistant from its three line edges.)

Solution

To do

Problem 8.4

For $n \in \mathbb{N}$, determine the n-simplices of a given volume that maximise the radius of their inscribed n-sphere.

Solution

To do

Problem 8.5

Let h_1, h_2, h_3, h_4 be the lengths of the altitudes of a tetrahedron. Let O be an interior point of the tetrahedron. Let d_1, d_2, d_3, d_4 be the distances between O and the planes containing the faces of the tetrahedron. Show that $h_1^4 + h_2^4 + h_3^4 + h_4^4 \geq 2^{10} d_1 d_2 d_3 d_4$.

(TODO: check how and if this generalises to higher dimensions)

Solution

To do

Problem 8.6

Prove that the heights of a tetrahedron are concurrent if and only if one of the heights has its base in the orthocenter of the corresponding face.

Solution

To do

Problem 8.7

Let $n \in \mathbb{N}$. Prove that in any n-simplex, the circumradius R and the inradius r are such that $R \geq nr$.
When is equality attained?
What is the set of attainable values of $\frac{R}{r}$?

Solution

To do

Problem 8.8

Let $n \in \mathbb{N}$. Given an n-simplex of unit volume, and one point on each of its sides, cut off corners from each vertex using the given points on the sides exiting that vertex. Prove that the total volume of the cutoff part is less than or equal to $\frac{n+1}{2^n}$.

Solution

To do

Problem 8.9

Let $ABCD$ be a tetrahedron such that ABC is equilateral and $\widehat{BAD} \cong \widehat{ACD} \cong \widehat{BCD}$. Prove that $ABCD$ is a regular pyramid on the base ABC; that is: prove that $\overline{AD} \cong \overline{BD} \cong \overline{CD}$.

Solution

To do

Problem 8.10

Prove that if the faces of a tetrahedron all have the same area, then they are congruent.
(TODO: generalise to arbitrary dimension.)

Solution

To do

Problem 8.11

Let $ABCD$ be a tetrahedron. Let O be a point on the face ABC. Prove that:

$$\frac{1}{2}\left(\widehat{ADB} + \widehat{BDC} + \widehat{CDA}\right) < \widehat{ODA} + \widehat{ODB} + \widehat{ODC} < \widehat{ADB} + \widehat{BDC} + \widehat{CDA}$$

Solution

To do

Problem 8.12

Prove that the sum of the measures of all dihedral angles of a tetrahedron is greater than 2π and less than 3π, and that for any value in that range there exists a tetrahedron that achieves it.

Solution

To do

Problem 8.13

If a tetrahedron is contained inside another tetrahedron, then is the sum of the lengths of the sides of the inner one less than that of the outer one? Is the sum of the areas of the faces of the inner tetrahedron less than that of the outer one?

Solution

To do

Problem 8.14

A regular tetrahedron $ABCD$ with side length a has its vertices on the surface of a double-cone whose vertex angle is $\frac{\pi}{2}$. The side $\overline{AB}$ lies on a generator of the cone. Determine the distance from the vertex of the cone to the line CD.

Solution

To do

Problem 8.15

Can a cube be inside a half-cone, with 7 vertices on the surface of the cone?

Solution

To do

Problem 8.16

Determine the distance between a circle inscribed in a face of a cube and a circle circumscribed about an adjacent face of the cube.
(TODO: investigate generalisations to higher dimensions.)

Solution

To do

Problem 8.17

Prove that if all the faces of a convex polyhedron are triangles, then there is an edge such that the angles that it forms with its adjacent co-facial edges are all acute.
(TODO: investigate if there are generalisations to higher dimensions.)

Solution

To do

Problem 8.18

Prove that an irregular octahedron is completely contained in the union of the balls that have its edges as diameters.

Solution

To do

Problem 8.19

Determine whether it is possible for a planar section of a rectangular parallelepiped to be an equilateral (or regular?) pentagon.

Solution

To do

Problem 8.20

Determine whether for any trihedral angle there exists a plane that intersects it in an equilateral triangle.
(TODO: what about in higher dimensions, with more lines, intersecting a hyperplane in regular simplices?)

Solution

To do

Problem 8.21

For $n \in \mathbb{N}$, determine the largest $a_n \in \mathbb{R}$ such that any convex n-polyhedron of n-volume 1 contains an n-simplex of n-volume a_n.

Solution

To do

Geometric constructions

Problem 9.1

Let ABC be a triangle. Using only straightedge and compass, construct a point $P \in \overline{AB}$ and a point $Q \in \overline{BC}$ such that $\overline{AP} \cong \overline{PQ} \cong \overline{QC}$.

Solution

To do

Problem 9.2

Prove that any two quadrilaterals are congruent if and only if their internal angles are congruent, in order, and their diagonals are congruent, in order.

Using only straightedge and compass, construct a quadrilateral, given its angles, in order and its diagonals, in order.

Solution

To do

Problem 9.3

Prove that any two quadrilaterals are congruent if and only if their sides are congruent, in order, and the segments between the midpoints of their first and third sides are congruent.

Using only straightedge and compass, reconstruct a quadrilateral, given segments congruent to its four sides, in order, and a segment congruent to the segment between the midpoints of the first and third sides.

Solution

To do

Problem 9.4

Given a point and an angle on a plane, construct, using only straightedge and compass, a line through the point that cuts the angle into a triangle of minimum perimeter.
Additionally, given also a segment, construct a line through the point that cuts the angle into a triangle whose perimeter is the length of the segment.

Solution

To do

Problem 9.5

Given a circle and one of its diameters, and given a point on the plane that does not lie on the circle nor on the line containing the diameter, construct, using only a straightedge, the perpendicular from the given point to the given diameter.

Solution

To do

Problem 9.6

Given a circle and one of its diameters, and given a point on the circle, distinct from the endpoints of the diameter, construct, using only a straightedge, the perpendicular from the given point to the given diameter.

Solution

To do

Problem 9.7

Given a segment and a positive integer n, divide the segment into n parts of equal lengths, using only a compass.

Solution

To do

Problem 9.8

Given two parallel segments and a positive integer n, divide one of the segments into n parts of equal lengths, using only a straightedge.

Solution

To do

Problem 9.9

Determine for which $n, k \in \mathbb{Z}^+$ it is possible, given k segments of lengths $a_1, \ldots, a_k$ on the plane, to construct, using only straightedge and compass, a segment of length b such that:

$$\sqrt[n]{b} = \sum_{i=1}^{k} \sqrt[n]{a_i}$$

Note: a segment of length 1 is not given.

Solution

Probably for $k = 1$ and any n; and for n a power of 2 and any k. To do this, use repeated geometric mean constructions (via the semicircle construcion, for example).

(TODO: complete the solution)

Problem 9.10

Reconstruct a square given one point from each side, using only straightedge and compass.
(Is such a square uniquely determined? Under what conditions? Maybe if the given points are not themselves vertices of a square?)
(What about a point on each of the lines containing the sides?)
(What about other polygons? What about solids in higher dimensions?)

Solution

To do

Problem 9.11

Using only straightedge and compass, construct the directrix and focus of a given parabola.
(What about similar problem for other conics?)

Solution

To do

Problem 9.12

Using only straightedge and compass, construct the center of a given sphere.
(What about higher dimensions?)

Solution

To do

Other

Problem 10.1

Let R, S, T be sets, with $\#(R) \geq 2$, $\#(S) \geq 2$, and $\#(T) \geq 3$. Let $f : R \times S \to T$ be such that $\#(\mathrm{im}(f)) \geq 3$. The elements $a \in R$ and $b \in S$ are such that the functions $S \to T : y \mapsto f(a, y)$ and $R \to T : x \mapsto f(x, b)$ are not constant.
Prove that there exist $p, r \in R$ and $q, s \in S$ such that $f(p, q)$, $f(r, q)$, $f(p, s)$ are all distinct.

Solution

Since the function $S \to T : y \mapsto f(a, y)$ is not constant, there exists $c \in S$ such that $f(a, c) \neq f(a, b)$. And since the function $R \to T : x \mapsto f(x, b)$ is not constant, there exists $d \in R$ such that $f(d, b) \neq f(a, b)$.

Now, if c and d can be chosen in such a way that $f(b, d) \neq f(a, c)$, then $p = a$, $q = b$, $r = d$, $s = t$ satisfy the requirements.
Otherwise, it means that for every $(x, y) \in R \times S \setminus \{(a, b)\}$ $f(x, b) = f(a, y) = v$ (for some $v \in T$). Since $\#(\mathrm{im}(f)) \geq 3$, there exists $(r, q) \in R \times S$ such that $f(r, q)$, v, and $f(a, b)$ are all distinct. Necessarily, then, $r \neq a$ and $q \neq b$. Now, the requirements are satisfied by taking $(p, s) = (a, b)$.

Problem 10.2

Let $f : \mathbb{R} \to \mathbb{R}$ be a function, with $f(0) = 0$, $f(1) = 1$, $f(88) = \sqrt{2}$. Prove that there exist $x, y \in \mathbb{R}$ with $|x - y| \leq 1$ such that $f(x + 1) > f(x)$ and $f(y + 2^n) \neq f(y)$ for some $n \in \mathbb{N}$.

Solution

To do