Coffin problems

Problems

Evaluations

Problem 1.1

Let $a, r \in \mathbb{C}$. For $n \in \mathbb{N}$, if $\cos(a + kr) \neq 0$ for all integers $0 \leq k \leq n$, evaluate the sum:

$$\sum_{k=0}^{n-1} \frac{1}{\cos(a + kr)\cos(a + (k+1)r)}$$

Problem 1.2

Find $\sin(1°)$, $\cos(1°)$, $\tan(1°)$, and $\sin(2°)$, $\cos(2°)$, $\tan(2°)$.

Problem 1.3

Evaluate $\tan\left(\frac{1}{7}\pi\right) \cdot \tan\left(\frac{3}{7}\pi\right) \cdot \tan\left(\frac{5}{7}\pi\right)$.

Problem 1.4

Determine the number of digits of 125^{100} in base 10.

Comparisons

Problem 2.1

Determine which among $\log_2(3)$ and $\log_3(5)$ is largest.

Problem 2.2

For $N \in \mathbb{N}$, determine which among $\displaystyle\prod_{n=2}^{N} \log_3(2n)$ and $\displaystyle 2\prod_{n=2}^{N} \log_3(2n-1)$ is largest.
(What about with a different base?)

Problem 2.3

Determine which among $\dfrac{8}{27}\pi$ and $\sin\left(\dfrac{8}{7}\right)$ is largest.

Problem 2.4

Determine which among $\sqrt[3]{413}$ and $6 + \sqrt[3]{3}$ is largest.

Problem 2.5

Prove that:

$$\sqrt[3]{3+\sqrt[3]{3}} + \sqrt[3]{3-\sqrt[3]{3}} < 2\sqrt[3]{3}$$

Problem 2.6

Prove that:

$$\sqrt{3+32\sin^2(15°)} + \cos(22°) + \cos(70°) + \cos(88°) + 2\sqrt{2}\sin(15°) > \frac{3}{2}\left(\cos(11°) + \cos(35°) + \cos(44°)\right)^2$$

Equations

Problem 3.1

For $a \in \mathbb{R}$, find all real numbers $x \geq -a$ that satisfy:

$$\sqrt{a + \sqrt{a + x}} = x$$

Problem 3.2

Find all $x \in \mathbb{R}$ that satisfy:

$$2\sqrt[3]{2x - 1} = x^3 + 1$$

Problem 3.3

Let $a \in \mathbb{R}^+$. Find all $x \in \mathbb{R}^+$ that satisfy:

$$x^{x^{\cdot^{\cdot^{\cdot^{x^a}}}}} = a$$

for a given height of the power tower.

Problem 3.4

Find all $x \in \mathbb{R}$ (or $\mathbb{C}$?) that satisfy:

$$x^4 - 14x^3 + 66x^2 - 115x + 66 + \frac{1}{4} = 0$$

Problem 3.5

Find all $x, y \in \mathbb{R}$ that satisfy:

$$\begin{cases} y \cdot (x + y)^2 = 9 \\ y \cdot (x^3 - y^3) = 7 \end{cases}$$

Problem 3.6

For each $n \in \mathbb{N}$, determine the set M_n of pairs $(a, b) \in \mathbb{R}^2$ such that the equation $x^2 - a = |x - b|$ has exactly n solutions in $\mathbb{R}$.

Describe the plot of each set M_n in $\mathbb{R}^2$.

Problem 3.7

Investigate the following equation, for $a, b \in \mathbb{R}$:

$$2^a + 2^{-a} = b\cos(\pi a)$$

Problem 3.8

Find all $x \in \mathbb{R}$ that satisfy:

$$\sin^7(x) + \frac{1}{\sin^3(x)} = \cos^7(x) + \frac{1}{\cos^3(x)}$$

(Are the numbers 3 and 7 important or could they be any?)

Problem 3.9

Find all $x \in \mathbb{R}$ that satisfy:

$$\sin^{\frac{11}{7}}(x) + \cos^{\frac{19}{11}}(x) = \sqrt{\frac{19}{7}}$$

(Are the numbers 11, 7, 19 important or could they be any?)

Problem 3.10

For $a, r \in \mathbb{R}$ with $a > 1$ and $\frac{1}{2} \leq r \leq \frac{1}{2}a$, find all $x \in \mathbb{R}$ that satisfy:

$$\left(1 - \frac{1}{a}\cos^2(x)\right)^r = \sin(x)$$

and describe the solution set when $r > \frac{1}{2}a$.

Problem 3.11

Find all $x \in \mathbb{R}$ (or $\mathbb{C}$?) that satisfy:

$$\cot(x) = \sin\left(x + \frac{\pi}{4}\right)$$

Problem 3.12

Find all $x \in \mathbb{R}$ that satisfy:

$$\sin^3(x)\cos\left(\frac{x}{2}\right) + \frac{1}{2}\sin(x)\sin\left(\frac{x}{2}\right)\left(1 + 2\cos\left(\frac{x}{2}\right)\right) - 6\sin^2\left(\frac{x}{2}\right) - 1 = 0$$

Problem 3.13

Find all $x \in \mathbb{R}^+$ that satisfy:

$$\frac{1}{16^x} = \log_{\frac{1}{16}}(x)$$

Inequations

Problem 4.1

Find all $x \in [-1, 1]$ that satisfy:

$$x \cdot \left(8\sqrt{1-x} + \sqrt{1+x}\right) \leq 11\sqrt{1+x} - 16\sqrt{1-x}$$

Problem 4.2

Find all $a \in \mathbb{R}$ such that for any $x \in \mathbb{R}^+$ the following holds:

$$ax^2 + 2x > 3a - 1$$

Problem 4.3

Find all $x \in \mathbb{R}$ that satisfy:

$$2^{\sin(x)} + 2^{\cos(x)} \geq 2^{1-\frac{1}{\sqrt{2}}}$$

Problem 4.4

Find all $x \in \mathbb{R}$ that satisfy:

$$\frac{1}{\sin^2(x)} < \frac{1}{x^2} + 1 - \frac{4}{\pi^2}$$

And determine

$$\lim_{x\to 0} \frac{1}{\sin^2(x)} - \frac{1}{x^2}$$

Problem 4.5

Determine the largest $a \in \mathbb{R}^+$ such that for all $x \in \left(0, \frac{\pi}{2}\right]$ the following holds:

$$\operatorname{sinc}^a(x) > \cos(x)$$

Problem 4.6

Find all $(x, y) \in (-3, 3) \times \mathbb{R}$ that satisfy:

$$3^y \log_3(9 - x^2) \leq 1 + 3^{2y}$$

Algebra and Number Theory

Problem 5.1

Prove that for any $\alpha \in \mathbb{R}$, α is irrational if and only if the set $\{n + m\alpha \mid n, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}$.

Problem 5.2

For each $a \in \mathbb{Z}$, let $\mathrm{P}(a)$ be the set of prime divisors of a. Characterise the set:

$$S = \left\{(a, b) \in \mathbb{N}^2 \;\middle|\; \mathrm{P}(a) = \mathrm{P}(b),\ \mathrm{P}(a+1) = \mathrm{P}(b+1)\right\}$$

What about the set $T = \left\{(a, b) \in \mathbb{Z}^2 \mid \mathrm{P}(a) = \mathrm{P}(b),\ \mathrm{P}(a+1) = \mathrm{P}(b+1)\right\}$?

Problem 5.3

Prove that for every $n \in \mathbb{Z}^+$:

$$\prod_{\substack{1 \leq p \leq n \\ p \text{ prime}}} p \leq 4^{n-1}$$

Problem 5.4

Prove that $\sin(10°)$, $\cos(10°)$, and $\tan(10°)$ are irrational and algebraic, determine their algebraic degrees over $\mathbb{Q}$, and determine their minimal polynomials over $\mathbb{Q}$.

Problem 5.5

1. Do there exist rational numbers $a, b > 0$ such that a^b is rational?
2. Do there exist rational numbers $a, b > 0$ such that a^b is irrational?
3. Do there exist $a \in \mathbb{R}^+$ rational and $b \in \mathbb{R}^+$ irrational such that a^b is rational?
4. Do there exist $a \in \mathbb{R}^+$ rational and $b \in \mathbb{R}^+$ irrational such that a^b is irrational?
5. Do there exist $a \in \mathbb{R}^+$ irrational and $b \in \mathbb{R}^+$ rational such that a^b is rational?
6. Do there exist $a \in \mathbb{R}^+$ irrational and $b \in \mathbb{R}^+$ rational such that a^b is irrational?
7. Do there exist irrational numbers $a, b \in \mathbb{R}^+$ such that a^b is rational?
8. Do there exist irrational numbers $a, b \in \mathbb{R}^+$ such that a^b is irrational?

Problem 5.6

The digit expansion of a number $a \in (0, 1)$ has 0 as first digit, then for every $n \in \mathbb{N}$, the digits $(2^n + 1)$-th to 2^{n+1}-th are the opposite of the digits 1-st to 2^n-th, respectively, where the opposite of the digit 1 is the digit 0, and viceversa. Prove that a is irrational.

Problem 5.7

For which integers $n \geq 1$ does there exist a regular n-gon in $\mathbb{R}^2$ whose vertices are rational points? (That is, whose vertices are in $\mathbb{Q}^2$).

(What about in $\mathbb{R}^m$?) (What about regular solids in higher dimensions?)

Problem 5.8

A square of side length 1 is given on the plane. Does there exist a point on the plane whose distances to the vertices of the square are all rational?
(What about other polygons?) (What about regular solids in any dimension?) (What is the maximum number of distances that can be rational?)

Problem 5.9

What is the largest cardinal α such that there exists a set $S \subseteq \mathbb{R}^2$ with $\#(S) = \alpha$, no three elements of which are collinear, and such that for every $p, q \in S$ the distance $\mathrm{d}(p, q)$ is integer?
(What about in higher dimension?)

Problem 5.10

Determine all pairs of positive rational numbers (a, b) such that $a^b = b^a$.

Problem 5.11

Determine all the pairs $(a, b) \in \mathbb{N}^2$ such that $a^2 + (a+1)^2 = b^2$.

Problem 5.12

Determine all $x, y \in \mathbb{Q}\left[\sqrt{2}\right]$ such that

$$x^2 + y^2 = 5 + 4\sqrt{2}$$

Problem 5.13

Rationalise the denominator in the following fraction:

$$\frac{1}{\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}}$$

Equivalently:
find two non-zero polynomials $s(x, y, z)$, $t(x, y, z)$ such that $t(x^3, y^3, z^3) = (x + y + z) \cdot s(x, y, z)$.

Analysis

Problem 6.1

Let $f : \mathbb{R} \to \mathbb{R}$ be a monotonically increasing function. Let $a, b \in \mathbb{R}$ with $a < b$. Determine which points $c \in (a, b)$ minimize the value:

$$\int_a^c f(x) - f(a) \,\mathrm{d}x + \int_c^b f(b) - f(x) \,\mathrm{d}x$$

Problem 6.2

Let $f : [0, 1] \to \mathbb{R}^+$ be a continuous function. Let $a, b \in \mathbb{R}^+$ such that $a \le f \le b$. Prove that:

$$ab \int_0^1 \frac{1}{f(x)} \,\mathrm{d}x \le a + b - \int_0^1 f(x) \,\mathrm{d}x$$

Problem 6.3

Let $a > 1$ be a real number. Let $I \subseteq \mathbb{R}$ be an interval. Find all functions $f : I \to \mathbb{R}$ such that for any $x, y \in \mathbb{R}$ the following holds:

$$f(x) - f(y) \le |x - y|^a$$

Problem 6.4

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for any $x \in \mathbb{R}$ the following holds:

$$f(f(x)) = x^2 - 2$$

Problem 6.5

Let $(a_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}$. If $\lim\limits_{n \to +\infty} (a_{n+1} - a_n) = 0$, then does $\lim\limits_{n \to +\infty} a_n$ exist (finite or infinite)?

Problem 6.6

Determine whether the following series converges. If it does, determine its value and its rate of convergence.

$$\sum_{n=1}^{+\infty} \frac{1}{n^3 + 3n^2 + 2n}$$

Problem 6.7

For $a, b, c, d \in \mathbb{R}$, what is the minimum value of $(a-d)^2 + (b-c)^2$ under the constraints $a^2 + 4b^2 = 4$ and $cd = 4$? And when is the minimum achieved?

Plane Geometry

Problem 7.1

Let ABC be a triangle, with $\widehat{ABC} = 80°$. Let O be a point inside ABC such that $\widehat{OAC} = 10°$ and $\widehat{OCA} = 30°$.
Express the angle $\widehat{ABO}$ in terms of $\frac{OB}{AC}$.

Problem 7.2

What is the maximum area for a triangle whose angle bisectors are all less than or equal to 1 in length? And when is the maximum achieved?
(What about the minimum area of a triangle whose bisectors are longer than or equal to 1?)

Problem 7.3

Four circles on a plane are mutually tangent to each other. The points of tangency are all distinct. Three of the circles have collinear centers. Determine the distance between the center of fourth circle and the line through the centers of the others, in terms of the radius of the fourth circle.
(There are two cases: one for internal tangency and one for external tangency)
(What about for spheres in higher dimension?)

Problem 7.4

Let ABC be a triangle. Let γ be its circumcircle. Let $\alpha_1, \alpha_2, \alpha_3$ be circles such that α_1 is tangent to $\overline{BC}$, to $\overline{CA}$, and to γ; α_2 is tangent to $\overline{AB}$, to $\overline{CA}$, and to γ; α_3 is tangent to $\overline{AB}$, to $\overline{BC}$, and to γ. Determine the radius of γ, given the radii of $\alpha_1, \alpha_2, \alpha_3$.

Distinguish all combinations of internal and external tangency between the circles.

Alernative formulation.

Three circles are each tangent to a (distinct) unordered pair of (distinct) sides of a triangle and to the circumcircle of the triangle. Determine the radius of the circumcircle, given the radii of the three circles.

Distinguish all combinations of internal and external tangency between the circles.

Problem 7.5

Let ABC be an equilateral triangle, and let O be a point inside it. Show that the lengths $\overline{AO}$, $\overline{BO}$, $\overline{CO}$ can be the side lengths of a triangle, and determine the measures of the internal angles in such triangle, in terms of $\widehat{AOB}$, $\widehat{BOC}$ and $\widehat{COA}$.
(What about in arbitrary dimensional simplex?)

Problem 7.6

Prove that a quadrilateral $ABCD$ is a rhombus if and only if the triangles AOB, BOC, COD, DOA are isoperimetetric, where O is the intersection of the diagonal lines AC and BD.

Problem 7.7

Given a triangle, let R be the radius of its circumscribed circle, and r the radius of its inscribed circle. Determine the distance s between the centers of the two circles.

Determine the set of possible values of s among all triangles that have a fixed circumradius R; when are the extremes achieved?
(What about in arbitrary-dimensional simplex?)

Problem 7.8

Given two intersecting lines r, s on a plane, and a real number $a \geq 0$, find the locus of points P of the plane such that $\mathrm{d}(P, r) + \mathrm{d}(P, s) = a$.
(What about in arbitrary dimension?) (What about intersection of three planes? What about $n+1$ n-hyperplanes?)

Problem 7.9

Prove that the area of a quadrilateral with side lengths a, b, c, d which admits both inscribed and circumscribed circles is $\sqrt{abcd}$.
(What about the viceversa?)

Problem 7.10

Determine the shortest networks that connect the four vertices of a square to each other.
(What about other configurations of points?)

Problem 7.11

For any partition U of $\mathbb{R}^2$, let $D_U = \{a \in \mathbb{R}^+ \mid \exists\, T \in U \text{ such that } \exists\, p, q \in T \text{ with } \mathrm{d}(p,q) = a\}$. What is the largest cardinal α such that for every partition U of $\mathbb{R}^2$ with $\#(U) = \alpha$, the set D_U is the whole $\mathbb{R}^+$?

(What about partitions of $\mathbb{R}^m$?)

Problem 7.12

Prove that a quadrilateral is cyclic if and only if the perpendiculars to each side passing through the midpoint of the opposite side are concurrent.

Problem 7.13

Determine the quadrilateral with the largest area, given the lengths of its four sides, in order.

Problem 7.14

Let $\overline{AB}$ be a chord in a circle, and let M be its midpoint. Let $\overline{CD}$ and $\overline{EF}$ be two other chords in the circle that pass through the point M, with C and F on opposite sides of $\overline{AB}$. Prove that $\overline{CF}$ intersects $\overline{AB}$ at a point P, and $\overline{ED}$ intersects $\overline{AB}$ at a point Q, on opposite sides of M, such that $\overline{MP} \cong \overline{MQ}$.

And prove that one and only one of the following holds:

- $\overline{CE}$ and $\overline{DF}$ are parallel to $\overline{AB}$;
- the line CE intersects the line AB at a point P', and the line DF intersects the line AB at a point Q', which are on opposite sides of M, such that $\overline{MP'} \cong \overline{MQ'}$.

(TODO: state the problem in projective geometry.)

Problem 7.15

Given a triangle, determine a line that halves both its area and its perimeter.
(How many such lines are there in a given triangle?) (Can the construction be carried out with straightedge and compass?)

Problem 7.16

Let $ABCD$ be a trapezoid with bases $\overline{AB}$ and $\overline{CD}$. Given a point $P \in \overline{AB}$, determine two points $Q_1, Q_2 \in \overline{CD}$ that, respectively, maximize the area of the quadrilateral intersection of the triangles ABQ_1 and CDP, and minimize the area of the quadrilateral intersection of the triangles ABQ_2 and CDP.

Problem 7.17

Let a, b, c be the side lengths of a triangle, and let α, β, γ be the measures of their opposite angles, respectively. Prove that:

$$\frac{b+c-2a}{\sin\left(\frac{\alpha}{2}\right)} + \frac{c+a-2b}{\sin\left(\frac{\beta}{2}\right)} + \frac{a+b-2c}{\sin\left(\frac{\gamma}{2}\right)} \geq 0$$

Problem 7.18

Let α, β, γ be the measures of the internal angles in a triangle. What is the maximum value of the quantity $\sqrt{\sin(\alpha)} + \sqrt{\sin(\beta)} + \sqrt{\sin(\gamma)}$, and when is it achieved?

Problem 7.19

Let a, b, c be the side lengths of a triangle, and let α, β, γ be the measures of their opposite angles, respectively. What are the minimum and maximum values of $\frac{a \cdot \alpha + b \cdot \beta + c \cdot \gamma}{a+b+c}$, and when are they achieved?

Problem 7.20

For a point P inside an equilateral (or regular?) hexagon of side length 1, what is the maximum sum of the distances between P and the vertices of the hexagon, and when is it achieved?
(What about in other polygons?)

Problem 7.21

A circle is given on a plane. Given two points on the plane, construct a circle that passes through those two points and is tangent to the first circle.
(Is the problem asking for a straightedge and compass construction?)
(What about for spheres in arbitrary dimension, and an appropriate number of points?)

Problem 7.22

Prove that if a triangle and a square are circumscribed about the same circle, then the portion of the square contained inside the triangle makes up more than half of the perimeter of the square.
(Note that the triangle is generic: it is not necessarily equilateral).

Problem 7.23

Let ABC be a triangle. Let M be the midpoint of $\overline{AC}$. Let $\overline{CL}$ be the angle bisector of $\widehat{BCA}$, with $L \in \overline{AB}$. Let P be the intersection point of $\overline{CL}$ and $\overline{BM}$. Prove that $\frac{\overline{CP}}{\overline{PL}} - \frac{\overline{AC}}{\overline{CB}} = 1$.

Problem 7.24

How many unordered pairs of triangles have as union a given quadrilateral?

Problem 7.25

n segments are given on the plane. Prove that the number of triangles whose sides are among those segments is $O\big(n^{3/2}\big)$.

Problem 7.26

Determine the largest $a \in \mathbb{R}^+$ such that every closed convex subset of $\mathbb{R}^2$ of area 1 contains a triangle of area a.

Problem 7.27

Let $ABCDE$ be a convex pentagon. The triangles ABC, BCD, CDE, DEA, EAB all have area 1. Determine the area of the pentagon.

Solid Geometry

Problem 8.1

Prove that if a sphere is tangent to all the edges of a three-dimensional quadrilateral, then the points of tangency are coplanar.
(Check if and how this generalises to higher dimensions.)

Problem 8.2

Show that if in a tetrahedron the sums of lengths of opposite edges are all equal, then the sums of opposite dihedral angles are all equal.

Problem 8.3

Find an equivalent condition for the bisectors of two trihedral angles of a tetrahedron to intersect.
(Note: a bisector of a trihedral angle is the locus of points that are equidistant from its three line edges.)

Problem 8.4

For $n \in \mathbb{N}$, determine the n-simplices of a given volume that maximise the radius of their inscribed n-sphere.

Problem 8.5

Let h_1, h_2, h_3, h_4 be the lengths of the altitudes of a tetrahedron. Let O be an interior point of the tetrahedron. Let d_1, d_2, d_3, d_4 be the distances between O and the planes containing the faces of the tetrahedron. Show that $h_1^4 + h_2^4 + h_3^4 + h_4^4 \geq 2^{10} d_1 d_2 d_3 d_4$.

(TODO: check how and if this generalises to higher dimensions)

Problem 8.6

Prove that the heights of a tetrahedron are concurrent if and only if one of the heights has its base in the orthocenter of the corresponding face.

Problem 8.7

Let $n \in \mathbb{N}$. Prove that in any n-simplex, the circumradius R and the inradius r are such that $R \geq nr$.
When is equality attained?
What is the set of attainable values of $\frac{R}{r}$?

Problem 8.8

Let $n \in \mathbb{N}$. Given an n-simplex of unit volume, and one point on each of its sides, cut off corners from each vertex using the given points on the sides exiting that vertex. Prove that the total volume of the cutoff part is less than or equal to $\frac{n+1}{2^n}$.

Problem 8.9

Let $ABCD$ be a tetrahedron such that ABC is equilateral and $\widehat{BAD} \cong \widehat{ACD} \cong \widehat{BCD}$. Prove that $ABCD$ is a regular pyramid on the base ABC; that is: prove that $\overline{AD} \cong \overline{BD} \cong \overline{CD}$.

Problem 8.10

Prove that if the faces of a tetrahedron all have the same area, then they are congruent.
(TODO: generalise to arbitrary dimension.)

Problem 8.11

Let $ABCD$ be a tetrahedron. Let O be a point on the face ABC. Prove that:

$$\frac{1}{2}\left(\widehat{ADB} + \widehat{BDC} + \widehat{CDA}\right) < \widehat{ODA} + \widehat{ODB} + \widehat{ODC} < \widehat{ADB} + \widehat{BDC} + \widehat{CDA}$$

Problem 8.12

Prove that the sum of the measures of all dihedral angles of a tetrahedron is greater than 2π and less than 3π, and that for any value in that range there exists a tetrahedron that achieves it.

Problem 8.13

If a tetrahedron is contained inside another tetrahedron, then is the sum of the lengths of the sides of the inner one less than that of the outer one? Is the sum of the areas of the faces of the inner tetrahedron less than that of the outer one?

Problem 8.14

A regular tetrahedron $ABCD$ with side length a has its vertices on the surface of a double-cone whose vertex angle is $\frac{\pi}{2}$. The side $\overline{AB}$ lies on a generator of the cone. Determine the distance from the vertex of the cone to the line CD.

Problem 8.15

Can a cube be inside a half-cone, with 7 vertices on the surface of the cone?

Problem 8.16

Determine the distance between a circle inscribed in a face of a cube and a circle circumscribed about an adjacent face of the cube.
(TODO: investigate generalisations to higher dimensions.)

Problem 8.17

Prove that if all the faces of a convex polyhedron are triangles, then there is an edge such that the angles that it forms with its adjacent co-facial edges are all acute.
(TODO: investigate if there are generalisations to higher dimensions.)

Problem 8.18

Prove that an irregular octahedron is completely contained in the union of the balls that have its edges as diameters.

Problem 8.19

Determine whether it is possible for a planar section of a rectangular parallelepiped to be an equilateral (or regular?) pentagon.

Problem 8.20

Determine whether for any trihedral angle there exists a plane that intersects it in an equilateral triangle. (TODO: what about in higher dimensions, with more lines, intersecting a hyperplane in regular simplices?)

Problem 8.21

For $n \in \mathbb{N}$, determine the largest $a_n \in \mathbb{R}$ such that any convex n-polyhedron of n-volume 1 contains an n-simplex of n-volume a_n.

Geometric constructions

Problem 9.1

Let ABC be a triangle. Using only straightedge and compass, construct a point $P \in \overline{AB}$ and a point $Q \in \overline{BC}$ such that $\overline{AP} \cong \overline{PQ} \cong \overline{QC}$.

Problem 9.2

Prove that any two quadrilaterals are congruent if and only if their internal angles are congruent, in order, and their diagonals are congruent, in order.

Using only straightedge and compass, construct a quadrilateral, given its angles, in order and its diagonals, in order.

Problem 9.3

Prove that any two quadrilaterals are congruent if and only if their sides are congruent, in order, and the segments between the midpoints of their first and third sides are congruent.

Using only straightedge and compass, reconstruct a quadrilateral, given segments congruent to its four sides, in order, and a segment congruent to the segment between the midpoints of the first and third sides.

Problem 9.4

Given a point and an angle on a plane, construct, using only straightedge and compass, a line through the point that cuts the angle into a triangle of minimum perimeter.
Additionally, given also a segment, construct a line through the point that cuts the angle into a triangle whose perimeter is the length of the segment.

Problem 9.5

Given a circle and one of its diameters, and given a point on the plane that does not lie on the circle nor on the line containing the diameter, construct, using only a straightedge, the perpendicular from the given point to the given diameter.

Problem 9.6

Given a circle and one of its diameters, and given a point on the circle, distinct from the endpoints of the diameter, construct, using only a straightedge, the perpendicular from the given point to the given diameter.

Problem 9.7

Given a segment and a positive integer n, divide the segment into n parts of equal lengths, using only a compass.

Problem 9.8

Given two parallel segments and a positive integer n, divide one of the segments into n parts of equal lengths, using only a straightedge.

Problem 9.9

Determine for which $n, k \in \mathbb{Z}^+$ it is possible, given k segments of lengths $a_1, \dots, a_k$ on the plane, to construct, using only straightedge and compass, a segment of length b such that:

$$\sqrt[n]{b} = \sum_{i=1}^{k} \sqrt[n]{a_i}$$

Note: a segment of length 1 is not given.

Problem 9.10

Reconstruct a square given one point from each side, using only straightedge and compass.
(Is such a square uniquely determined? Under what conditions? Maybe if the given points are not themselves vertices of a square?)
(What about a point on each of the lines containing the sides?)
(What about other polygons? What about solids in higher dimensions?)

Problem 9.11

Using only straightedge and compass, construct the directrix and focus of a given parabola.
(What about similar problem for other conics?)

Problem 9.12

Using only straightedge and compass, construct the center of a given sphere.
(What about higher dimensions?)

Other

Problem 10.1

Let R, S, T be sets, with $\#(R) \geq 2$, $\#(S) \geq 2$, and $\#(T) \geq 3$. Let $f : R \times S \to T$ be such that $\#(\mathrm{im}(f)) \geq 3$. The elements $a \in R$ and $b \in S$ are such that the functions $S \to T : y \mapsto f(a, y)$ and $R \to T : x \mapsto f(x, b)$ are not constant.
Prove that there exist $p, r \in R$ and $q, s \in S$ such that $f(p, q)$, $f(r, q)$, $f(p, s)$ are all distinct.

Problem 10.2

Let $f : \mathbb{R} \to \mathbb{R}$ be a function, with $f(0) = 0$, $f(1) = 1$, $f(88) = \sqrt{2}$. Prove that there exist $x, y \in \mathbb{R}$ with $|x - y| \leq 1$ such that $f(x + 1) > f(x)$ and $f(y + 2^n) \neq f(y)$ for some $n \in \mathbb{N}$.