Coffin problems

26/05/2026

Introduction

This document presents a collection of mathematical problems known as *coffin problems*: they are challenging problems that in the 1960s, 1970s, and 1980s were given at some Soviet universities' admission exams with the goal of preventing the enrolment of students who were considered 'undesirable', category which mainly included people of Jewish origin. Such students were administered these problems at their oral examinations.

The solutions to the problems are not provided in this document.

The problems listed in this document are generalisations of the original problems.

The problems are categorised based on topic: each section lists problems that fall under a particular category.

The references from which the problems were sourced are listed at the end of the document. Moreover, at each problem statement, the specific sources in which that problem is documented are cited.

Contents

Problems

Evaluations

Problem 1.1

Source: [8, problem 9].

Let $a, r \in \mathbb{C}$. For $n \in \mathbb{N}$, if $\cos(a + kr) \neq 0$ for all integers $0 \leq k \leq n$, evaluate the sum:

$$\sum_{k=0}^{n-1} \frac{1}{\cos(a + kr)\cos(a + (k+1)r)}$$

Problem 1.2

Source: [8, problem 41].

Find $\sin(1°)$, $\cos(1°)$, $\tan(1°)$, and $\sin(2°)$, $\cos(2°)$, $\tan(2°)$.

Problem 1.3

Source: [9, problem 1].

Evaluate $\tan\left(\frac{1}{7}\pi\right) \cdot \tan\left(\frac{3}{7}\pi\right) \cdot \tan\left(\frac{5}{7}\pi\right)$.

Problem 1.4

Source: [10, problem 18]; [11, problem 12].

Determine the number of digits of 125^{100} in base 10.

Comparisons

Problem 2.1

Source: [10, problem 17]; [11, problem 11].

Determine which among $\log_2(3)$ and $\log_3(5)$ is largest.

Problem 2.2

Source: [14] and [23, problem 20].

For $N \in \mathbb{N}$, determine which among $\displaystyle\prod_{n=2}^{N} \log_3(2n)$ and $\displaystyle 2\prod_{n=2}^{N} \log_3(2n-1)$ is largest.
(What about with a different base?)

Problem 2.3

Source: [3, page 27].

Determine which among $\dfrac{8}{27}\pi$ and $\sin\left(\dfrac{8}{7}\right)$ is largest.

Problem 2.4

Source: [13, problem 1].

Determine which among $\sqrt[3]{413}$ and $6 + \sqrt[3]{3}$ is largest.

Problem 2.5

Source: [6, page 13, page 7 in the Russian version].

Prove that:

$$\sqrt[3]{3+\sqrt[3]{3}} + \sqrt[3]{3-\sqrt[3]{3}} < 2\sqrt[3]{3}$$

Problem 2.6

Source: rephrased from [14, Russian version, page 11].

Prove that:

$$\sqrt{3+32\sin^2(15°)} + \cos(22°) + \cos(70°) + \cos(88°) + 2\sqrt{2}\sin(15°) > \frac{3}{2}\left(\cos(11°) + \cos(35°) + \cos(44°)\right)^2$$

Equations

Problem 3.1

Source: [8, problem 8]; [3, page 23, problem 3].

For $a \in \mathbb{R}$, find all real numbers $x \geq -a$ that satisfy:

$$\sqrt{a + \sqrt{a + x}} = x$$

Problem 3.2

Source: [8, problem 12]; [10, problem 4]; [11, problem 2].

Find all $x \in \mathbb{R}$ that satisfy:

$$2\sqrt[3]{2x - 1} = x^3 + 1$$

Problem 3.3

Source: [8, problem 40].

Let $a \in \mathbb{R}^+$. Find all $x \in \mathbb{R}^+$ that satisfy:

$$x^{x^{\cdot^{\cdot^{\cdot^{x^a}}}}} = a$$

for a given height of the power tower.

Problem 3.4

Source: [14] and [23, problem 16].

Find all $x \in \mathbb{R}$ (or $\mathbb{C}$?) that satisfy:

$$x^4 - 14x^3 + 66x^2 - 115x + 66 + \frac{1}{4} = 0$$

Problem 3.5

Source: [14] and [23, problem 12].

Find all $x, y \in \mathbb{R}$ that satisfy:

$$\begin{cases} y \cdot (x+y)^2 = 9 \\ y \cdot (x^3 - y^3) = 7 \end{cases}$$

Problem 3.6

Source: [6, page 7, page 4 in the Russian version].

For each $n \in \mathbb{N}$, determine the set M_n of pairs $(a, b) \in \mathbb{R}^2$ such that the equation $x^2 - a = |x - b|$ has exactly n solutions in $\mathbb{R}$.

Describe the plot of each set M_n in $\mathbb{R}^2$.

Problem 3.7

Source: [8, problem 20].

Investigate the following equation, for $a, b \in \mathbb{R}$:

$$2^a + 2^{-a} = b\cos(\pi a)$$

Problem 3.8

Source: [8, problem 17]; [10, problem 5].

Find all $x \in \mathbb{R}$ that satisfy:

$$\sin^7(x) + \frac{1}{\sin^3(x)} = \cos^7(x) + \frac{1}{\cos^3(x)}$$

(Are the numbers 3 and 7 important or could they be any?)

Problem 3.9

Source: [6, page 13, page 7 in the Russian version].

Find all $x \in \mathbb{R}$ that satisfy:

$$\sin^{\frac{11}{7}}(x) + \cos^{\frac{19}{11}}(x) = \sqrt{\frac{19}{7}}$$

(Are the numbers 11, 7, 19 important or could they be any?)

Problem 3.10

Source: [8, problem 18].

For $a, r \in \mathbb{R}$ with $a > 1$ and $\frac{1}{2} \leq r \leq \frac{1}{2}a$, find all $x \in \mathbb{R}$ that satisfy:

$$\left(1 - \frac{1}{a}\cos^2(x)\right)^r = \sin(x)$$

and describe the solution set when $r > \frac{1}{2}a$.

Problem 3.11

Source: [6, page 10, page 5 in the Russian version, with typo].

Find all $x \in \mathbb{R}$ (or $\mathbb{C}$?) that satisfy:

$$\cot(x) = \sin\left(x + \frac{\pi}{4}\right)$$

Problem 3.12

Source: [6, page 7, page 4 in the Russian version].

Find all $x \in \mathbb{R}$ that satisfy:

$$\sin^3(x)\cos\left(\frac{x}{2}\right) + \frac{1}{2}\sin(x)\sin\left(\frac{x}{2}\right)\left(1 + 2\cos\left(\frac{x}{2}\right)\right) - 6\sin^2\left(\frac{x}{2}\right) - 1 = 0$$

Problem 3.13

Source: personal communications with Tanya Khovanova.

Find all $x \in \mathbb{R}^+$ that satisfy:

$$\frac{1}{16^x} = \log_{\frac{1}{16}}(x)$$

Inequations

Problem 4.1

Source: [8, problem 1]; [10, problem 1].

Find all $x \in [-1, 1]$ that satisfy:

$$x \cdot \left(8\sqrt{1-x} + \sqrt{1+x}\right) \leq 11\sqrt{1+x} - 16\sqrt{1-x}$$

Problem 4.2

Source: [14] and [23, problem 24].

Find all $a \in \mathbb{R}$ such that for any $x \in \mathbb{R}^+$ the following holds:

$$ax^2 + 2x > 3a - 1$$

Problem 4.3

Source: [8, problem 44].

Find all $x \in \mathbb{R}$ that satisfy:

$$2^{\sin(x)} + 2^{\cos(x)} \geq 2^{1-\frac{1}{\sqrt{2}}}$$

Problem 4.4

Source: [14, missing the interval in the English version] and [23, problem 6]; [13, problem 10 (with wrong inequality order)].

Find all $x \in \mathbb{R}$ that satisfy:

$$\frac{1}{\sin^2(x)} < \frac{1}{x^2} + 1 - \frac{4}{\pi^2}$$

And determine

$$\lim_{x\to 0} \frac{1}{\sin^2(x)} - \frac{1}{x^2}$$

Problem 4.5

Source: [13, problem 9].

Determine the largest $a \in \mathbb{R}^+$ such that for all $x \in \left(0, \frac{\pi}{2}\right]$ the following holds:

$$\operatorname{sinc}^a(x) > \cos(x)$$

Problem 4.6

Source: [6, page 9, page 5 in the Russian version].

Find all $(x, y) \in (-3, 3) \times \mathbb{R}$ that satisfy:

$$3^y \log_3\left(9 - x^2\right) \leq 1 + 3^{2y}$$

Algebra and Number Theory

Problem 5.1

Source: [8, problem 5].

Prove that for any $\alpha \in \mathbb{R}$, α is irrational if and only if the set $\{n + m\alpha \mid n, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}$.

Problem 5.2

Source: [8, problem 14]; [14] and [23, problem 4]; [3, page 23, problem 5]; [22].

For each $a \in \mathbb{Z}$, let $\mathrm{P}(a)$ be the set of prime divisors of a. Characterise the set:

$$S = \left\{(a, b) \in \mathbb{N}^2 \;\middle|\; \mathrm{P}(a) = \mathrm{P}(b),\ \mathrm{P}(a+1) = \mathrm{P}(b+1)\right\}$$

What about the set $T = \{(a, b) \in \mathbb{Z}^2 \mid \mathrm{P}(a) = \mathrm{P}(b),\ \mathrm{P}(a+1) = \mathrm{P}(b+1)\}$?

Problem 5.3

Source: [8, problem 34].

Prove that for every $n \in \mathbb{Z}^+$:

$$\prod_{\substack{1 \le p \le n \\ p \text{ prime}}} p \le 4^{n-1}$$

Problem 5.4

Source: [8, problem 38]; [10, problem 11]; [11, problem 7].

Prove that $\sin(10°)$, $\cos(10°)$, and $\tan(10°)$ are irrational and algebraic, determine their algebraic degrees over $\mathbb{Q}$, and determine their minimal polynomials over $\mathbb{Q}$.

Problem 5.5

Source: [9, problem 2].

1. Do there exist rational numbers $a, b > 0$ such that a^b is rational?
2. Do there exist rational numbers $a, b > 0$ such that a^b is irrational?
3. Do there exist $a \in \mathbb{R}^+$ rational and $b \in \mathbb{R}^+$ irrational such that a^b is rational?
4. Do there exist $a \in \mathbb{R}^+$ rational and $b \in \mathbb{R}^+$ irrational such that a^b is irrational?

5. Do there exist $a \in \mathbb{R}^+$ irrational and $b \in \mathbb{R}^+$ rational such that a^b is rational?
6. Do there exist $a \in \mathbb{R}^+$ irrational and $b \in \mathbb{R}^+$ rational such that a^b is irrational?
7. Do there exist irrational numbers $a, b \in \mathbb{R}^+$ such that a^b is rational?
8. Do there exist irrational numbers $a, b \in \mathbb{R}^+$ such that a^b is irrational?

Problem 5.6

Source: [9, problem 4].

The digit expansion of a number $a \in (0, 1)$ has 0 as first digit, then for every $n \in \mathbb{N}$, the digits $(2^n + 1)$-th to 2^{n+1}-th are the opposite of the digits 1-st to 2^n-th, respectively, where the opposite of the digit 1 is the digit 0, and viceversa. Prove that a is irrational.

Problem 5.7

Source: [10, problem 13]; [11, problem 8].

For which integers $n \geq 1$ does there exist a regular n-gon in $\mathbb{R}^2$ whose vertices are rational points? (That is, whose vertices are in $\mathbb{Q}^2$).

(What about in $\mathbb{R}^m$?) (What about regular solids in higher dimensions?)

Problem 5.8

Source: [16, page 36, page 28 in the new version].

A square of side length 1 is given on the plane. Does there exist a point on the plane whose distances to the vertices of the square are all rational?
(What about other polygons?) (What about regular solids in any dimension?) (What is the maximum number of distances that can be rational?)

Problem 5.9

Source: [8, problem 42]; [10, problem 12].

What is the largest cardinal α such that there exists a set $S \subseteq \mathbb{R}^2$ with $\#(S) = \alpha$, no three elements of which are collinear, and such that for every $p, q \in S$ the distance $\mathrm{d}(p, q)$ is integer?
(What about in higher dimension?)

Problem 5.10

Source: [13, problem 2].

Determine all pairs of positive rational numbers (a, b) such that $a^b = b^a$.

Problem 5.11

Source: [13, problem 5].

Determine all the pairs $(a, b) \in \mathbb{N}^2$ such that $a^2 + (a+1)^2 = b^2$.

Problem 5.12

Source: [13, problem 12].

Determine all $x, y \in \mathbb{Q}\left[\sqrt{2}\right]$ such that

$$x^2 + y^2 = 5 + 4\sqrt{2}$$

Problem 5.13

Source: [10, problem 19].

Rationalise the denominator in the following fraction:

$$\frac{1}{\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}}$$

Equivalently:
find two non-zero polynomials $s(x, y, z)$, $t(x, y, z)$ such that $t(x^3, y^3, z^3) = (x + y + z) \cdot s(x, y, z)$.

Analysis

Problem 6.1

Source: [9, problem 3]; [10, problem 21]; [11, problem 14].

Let $f : \mathbb{R} \to \mathbb{R}$ be a monotonically increasing function. Let $a, b \in \mathbb{R}$ with $a < b$. Determine which points $c \in (a, b)$ minimize the value:

$$\int_a^c f(x) - f(a)\,\mathrm{d}x + \int_c^b f(b) - f(x)\,\mathrm{d}x$$

Problem 6.2

Source: [6, page 10, page 5 in the Russian version].

Let $f : [0,1] \to \mathbb{R}^+$ be a continuous function. Let $a, b \in \mathbb{R}^+$ such that $a \le f \le b$. Prove that:

$$ab \int_0^1 \frac{1}{f(x)}\,\mathrm{d}x \le a + b - \int_0^1 f(x)\,\mathrm{d}x$$

Problem 6.3

Source: [8, problem 7]; [10, problem 2]; [11, problem 1].

Let $a > 1$ be a real number. Let $I \subseteq \mathbb{R}$ be an interval. Find all functions $f : I \to \mathbb{R}$ such that for any $x, y \in \mathbb{R}$ the following holds:

$$f(x) - f(y) \le |x - y|^a$$

Problem 6.4

Source: [16, page 24, (page 31 in the old version)].

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for any $x \in \mathbb{R}$ the following holds:

$$f(f(x)) = x^2 - 2$$

Problem 6.5

Source: [8, problem 37].

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. If $\lim\limits_{n\to+\infty}(a_{n+1}-a_n)=0$, then does $\lim\limits_{n\to+\infty}a_n$ exist (finite or infinite)?

Problem 6.6

Source: [14] and [23, problem 15].

Determine whether the following series converges. If it does, determine its value and its rate of convergence.

$$\sum_{n=1}^{+\infty}\frac{1}{n^3+3n^2+2n}$$

Problem 6.7

Source: [14, with typo in the English version] and [23, problem 8]; [13, problem 8].

For $a,b,c,d\in\mathbb{R}$, what is the minimum value of $(a-d)^2+(b-c)^2$ under the constraints $a^2+4b^2=4$ and $cd=4$? And when is the minimum achieved?

Plane Geometry

Problem 7.1

Source: [8, problem 6].

Let ABC be a triangle, with $\widehat{ABC} = 80°$. Let O be a point inside ABC such that $\widehat{OAC} = 10°$ and $\widehat{OCA} = 30°$.
Express the angle $\widehat{ABO}$ in terms of $\frac{\overline{OB}}{\overline{AC}}$.

Problem 7.2

Source: [8, problem 15].

What is the maximum area for a triangle whose angle bisectors are all less than or equal to 1 in length? And when is the maximum achieved?
(What about the minimum area of a triangle whose bisectors are longer than or equal to 1?)

Problem 7.3

Source: [8, problem 21].

Four circles on a plane are mutually tangent to each other. The points of tangency are all distinct. Three of the circles have collinear centers. Determine the distance between the center of fourth circle and the line through the centers of the others, in terms of the radius of the fourth circle.
(There are two cases: one for internal tangency and one for external tangency)
(What about for spheres in higher dimension?)

Problem 7.4

Source: [8, problem 24].

Let ABC be a triangle. Let γ be its circumcircle. Let $\alpha_1, \alpha_2, \alpha_3$ be circles such that α_1 is tangent to $\overline{BC}$, to $\overline{CA}$, and to γ; α_2 is tangent to $\overline{AB}$, to $\overline{CA}$, and to γ; α_3 is tangent to $\overline{AB}$, to $\overline{BC}$, and to γ. Determine the radius of γ, given the radii of $\alpha_1, \alpha_2, \alpha_3$.

Distinguish all combinations of internal and external tangency between the circles.

Alernative formulation.

Three circles are each tangent to a (distinct) unordered pair of (distinct) sides of a triangle and to the circumcircle of the triangle. Determine the radius of the circumcircle, given the radii of the three circles.

Distinguish all combinations of internal and external tangency between the circles.

Problem 7.5

Source: [8, problem 26]; [10, problem 8].

Let ABC be an equilateral triangle, and let O be a point inside it. Show that the lengths $\overline{AO}$, $\overline{BO}$, $\overline{CO}$ can be the side lengths of a triangle, and determine the measures of the internal angles in such triangle, in terms of $\widehat{AOB}$, $\widehat{BOC}$ and $\widehat{COA}$.
(What about in arbitrary dimensional simplex?)

Problem 7.6

Source: [8, problem 27].

Prove that a quadrilateral $ABCD$ is a rhombus if and only if the triangles AOB, BOC, COD, DOA are isoperimetetric, where O is the intersection of the diagonal lines AC and BD.

Problem 7.7

Source: adapted from [8, problem 29] and [9, problem 5].

Given a triangle, let R be the radius of its circumscribed circle, and r the radius of its inscribed circle. Determine the distance s between the centers of the two circles.

Determine the set of possible values of s among all triangles that have a fixed circumradius R; when are the extremes achieved?
(What about in arbitrary-dimensional simplex?)

Problem 7.8

Source: [8, problem 30]; [10, problem 9]; [11, problem 5].

Given two intersecting lines r, s on a plane, and a real number $a \geq 0$, find the locus of points P of the plane such that $\mathrm{d}(P, r) + \mathrm{d}(P, s) = a$.
(What about in arbitrary dimension?) (What about intersection of three planes? What about $n + 1$ n-hyperplanes?)

Problem 7.9

Source: [8, problem 33].

Prove that the area of a quadrilateral with side lengths a, b, c, d which admits both inscribed and circumscribed circles is $\sqrt{abcd}$.
(What about the viceversa?)

Problem 7.10

Source: [8, problem 35].

Determine the shortest networks that connect the four vertices of a square to each other.
(What about other configurations of points?)

Problem 7.11

Source: [8, problem 36].

For any partition U of $\mathbb{R}^2$, let $D_U = \{a \in \mathbb{R}^+ \mid \exists\, T \in U \text{ such that } \exists\, p, q \in T \text{ with } \mathrm{d}(p,q) = a\}$. What is the largest cardinal α such that for every partition U of $\mathbb{R}^2$ with $\#(U) = \alpha$, the set D_U is the whole $\mathbb{R}^+$?

(What about partitions of $\mathbb{R}^m$?)

Problem 7.12

Source: [20].

Prove that a quadrilateral is cyclic if and only if the perpendiculars to each side passing through the midpoint of the opposite side are concurrent.

Problem 7.13

Source: [10, problem 15]; [11, problem 9].

Determine the quadrilateral with the largest area, given the lengths of its four sides, in order.

Problem 7.14

Source: [14] and [23, problem 1].

Let $\overline{AB}$ be a chord in a circle, and let M be its midpoint. Let $\overline{CD}$ and $\overline{EF}$ be two other chords in the circle that pass through the point M, with C and F on opposite sides of $\overline{AB}$. Prove that $\overline{CF}$ intersects $\overline{AB}$ at a point P, and $\overline{ED}$ intersects $\overline{AB}$ at a point Q, on opposite sides of M, such that $\overline{MP} \cong \overline{MQ}$.

And prove that one and only one of the following holds:

- $\overline{CE}$ and $\overline{DF}$ are parallel to $\overline{AB}$;
- the line CE intersects the line AB at a point P', and the line DF intersects the line AB at a point Q', which are on opposite sides of M, such that $\overline{MP'} \cong \overline{MQ'}$.

(TODO: state the problem in projective geometry.)

Problem 7.15

Source: [14] and [23, problem 5]; [13, problem 7].

Given a triangle, determine a line that halves both its area and its perimeter.
(How many such lines are there in a given triangle?) (Can the construction be carried out with straightedge and compass?)

Problem 7.16

Source: maximisation problem: [14] and [23, problem 9]; minimisation problem: [6, page 7, page 4 in the Russian version].

Let $ABCD$ be a trapezoid with bases $\overline{AB}$ and $\overline{CD}$. Given a point $P \in \overline{AB}$, determine two points $Q_1, Q_2 \in \overline{CD}$ that, respectively, maximize the area of the quadrilateral intersection of the triangles ABQ_1 and CDP, and minimize the area of the quadrilateral intersection of the triangles ABQ_2 and CDP.

Problem 7.17

Source: [14] and [23, problem 13].

Let a, b, c be the side lengths of a triangle, and let α, β, γ be the measures of their opposite angles, respectively. Prove that:

$$\frac{b+c-2a}{\sin\left(\frac{\alpha}{2}\right)} + \frac{c+a-2b}{\sin\left(\frac{\beta}{2}\right)} + \frac{a+b-2c}{\sin\left(\frac{\gamma}{2}\right)} \geq 0$$

Problem 7.18

Source: [14, Russian version, page 5].

Let α, β, γ be the measures of the internal angles in a triangle. What is the maximum value of the quantity $\sqrt{\sin(\alpha)} + \sqrt{\sin(\beta)} + \sqrt{\sin(\gamma)}$, and when is it achieved?

Problem 7.19

Source: [14] and [23, problem 25].

Let a, b, c be the side lengths of a triangle, and let α, β, γ be the measures of their opposite angles, respectively. What are the minimum and maximum values of $\frac{a \cdot \alpha + b \cdot \beta + c \cdot \gamma}{a+b+c}$, and when are they achieved?

Problem 7.20

Source: [6, page 12, page 6 in the Russian version] (the Russian version says regular hexagon, while the English version says equilateral hexagon).

For a point P inside an equilateral (or regular?) hexagon of side length 1, what is the maximum sum of the distances between P and the vertices of the hexagon, and when is it achieved?
(What about in other polygons?)

Problem 7.21

Source: [4, Chapter 3, page 36].

A circle is given on a plane. Given two points on the plane, construct a circle that passes through those two points and is tangent to the first circle.
(Is the problem asking for a straightedge and compass construction?)
(What about for spheres in arbitrary dimension, and an appropriate number of points?)

Problem 7.22

Source: [3, page 23, problem 4].

Prove that if a triangle and a square are circumscribed about the same circle, then the portion of the square contained inside the triangle makes up more than half of the perimeter of the square.
(Note that the triangle is generic: it is not necessarily equilateral).

Problem 7.23

Source: [1, page 7, footnote 16].

Let ABC be a triangle. Let M be the midpoint of $\overline{AC}$. Let $\overline{CL}$ be the angle bisector of $\widehat{BCA}$, with $L \in \overline{AB}$. Let P be the intersection point of $\overline{CL}$ and $\overline{BM}$. Prove that $\frac{\overline{CP}}{\overline{PL}} - \frac{\overline{AC}}{\overline{CB}} = 1$.

Problem 7.24

Source: [14] and [23, problem 14].

How many unordered pairs of triangles have as union a given quadrilateral?

Problem 7.25

Source: [14, with typo in the English version] and [23, problem 22].

n segments are given on the plane. Prove that the number of triangles whose sides are among those segments is $O\big(n^{3/2}\big)$.

Problem 7.26

Source: [13, problem 3].

Determine the largest $a \in \mathbb{R}^+$ such that every closed convex subset of $\mathbb{R}^2$ of area 1 contains a triangle of area a.

Problem 7.27

Source: [13, problem 11].

Let $ABCDE$ be a convex pentagon. The triangles ABC, BCD, CDE, DEA, EAB all have area 1. Determine the area of the pentagon.

Solid Geometry

Problem 8.1

Source: [8, problem 31]; [10, problem 10]; [11, problem 6]; [14] and [23, problem 2].

Prove that if a sphere is tangent to all the edges of a three-dimensional quadrilateral, then the points of tangency are coplanar.
(Check if and how this generalises to higher dimensions.)

Problem 8.2

Source: [8, problem 2].

Show that if in a tetrahedron the sums of lengths of opposite edges are all equal, then the sums of opposite dihedral angles are all equal.

Problem 8.3

Source: [8, problem 3].

Find an equivalent condition for the bisectors of two trihedral angles of a tetrahedron to intersect.
(Note: a bisector of a trihedral angle is the locus of points that are equidistant from its three line edges.)

Problem 8.4

Source: [8, problem 4].

For $n \in \mathbb{N}$, determine the n-simplices of a given volume that maximise the radius of their inscribed n-sphere.

Problem 8.5

Source: [14] and [23, problem 11].

Let h_1, h_2, h_3, h_4 be the lengths of the altitudes of a tetrahedron. Let O be an interior point of the tetrahedron. Let d_1, d_2, d_3, d_4 be the distances between O and the planes containing the faces of the tetrahedron. Show that $h_1^4 + h_2^4 + h_3^4 + h_4^4 \geq 2^{10} d_1 d_2 d_3 d_4$.

(TODO: check how and if this generalises to higher dimensions)

Problem 8.6

Source: personal communications with Tanya Khovanova.

Prove that the heights of a tetrahedron are concurrent if and only if one of the heights has its base in the orthocenter of the corresponding face.

Problem 8.7

Source: [8, problem 13].

Let $n \in \mathbb{N}$. Prove that in any n-simplex, the circumradius R and the inradius r are such that $R \geq nr$. When is equality attained?
What is the set of attainable values of $\frac{R}{r}$?

Problem 8.8

Source: [14] and [23, problem 7]; [13, problem 13].

Let $n \in \mathbb{N}$. Given an n-simplex of unit volume, and one point on each of its sides, cut off corners from each vertex using the given points on the sides exiting that vertex. Prove that the total volume of the cutoff part is less than or equal to $\frac{n+1}{2^n}$.

Problem 8.9

Source: [8, problem 11].

Let $ABCD$ be a tetrahedron such that ABC is equilateral and $\widehat{BAD} \cong \widehat{ACD} \cong \widehat{BCD}$. Prove that $ABCD$ is a regular pyramid on the base ABC; that is: prove that $\overline{AD} \cong \overline{BD} \cong \overline{CD}$.

Problem 8.10

Source: [8, problem 32]; [14] and [23, problem 3].

Prove that if the faces of a tetrahedron all have the same area, then they are congruent.
(TODO: generalise to arbitrary dimension.)

Problem 8.11

Source: [6, page 13, page 7 in the Russian version].

Let $ABCD$ be a tetrahedron. Let O be a point on the face ABC. Prove that:

$$\frac{1}{2}\left(\widehat{ADB}+\widehat{BDC}+\widehat{CDA}\right) < \widehat{ODA}+\widehat{ODB}+\widehat{ODC} < \widehat{ADB}+\widehat{BDC}+\widehat{CDA}$$

Problem 8.12

Source: personal communications with Tanya Khovanova.

Prove that the sum of the measures of all dihedral angles of a tetrahedron is greater than 2π and less than 3π, and that for any value in that range there exists a tetrahedron that achieves it.

Problem 8.13

Source: personal communications with Tanya Khovanova.

If a tetrahedron is contained inside another tetrahedron, then is the sum of the lengths of the sides of the inner one less than that of the outer one? Is the sum of the areas of the faces of the inner tetrahedron less than that of the outer one?

Problem 8.14

Source: [14] and [23, problem 19].

A regular tetrahedron $ABCD$ with side length a has its vertices on the surface of a double-cone whose vertex angle is $\frac{\pi}{2}$. The side $\overline{AB}$ lies on a generator of the cone. Determine the distance from the vertex of the cone to the line CD.

Problem 8.15

Source: [14] and [23, problem 17].

Can a cube be inside a half-cone, with 7 vertices on the surface of the cone?

Problem 8.16

Source: [14] and [23, problem 21].

Determine the distance between a circle inscribed in a face of a cube and a circle circumscribed about an adjacent face of the cube.
(TODO: investigate generalisations to higher dimensions.)

Problem 8.17

Source: [8, problem 43].

Prove that if all the faces of a convex polyhedron are triangles, then there is an edge such that the angles that it forms with its adjacent co-facial edges are all acute.
(TODO: investigate if there are generalisations to higher dimensions.)

Problem 8.18

Source: [9, problem 6].

Prove that an irregular octahedron is completely contained in the union of the balls that have its edges as diameters.

Problem 8.19

Source: [8, problem 25].

Determine whether it is possible for a planar section of a rectangular parallelepiped to be an equilateral (or regular?) pentagon.

Problem 8.20

Source: [14] and [23, problem 10]; [6, page 9, page 5 in the Russian version].

Determine whether for any trihedral angle there exists a plane that intersects it in an equilateral triangle.
(TODO: what about in higher dimensions, with more lines, intersecting a hyperplane in regular simplices?)

Problem 8.21

Source: [13, problem 4].

For $n \in \mathbb{N}$, determine the largest $a_n \in \mathbb{R}$ such that any convex n-polyhedron of n-volume 1 contains an n-simplex of n-volume a_n.

Geometric constructions

Problem 9.1

Source: [8, problem 10]; [10, problem 3].

Let ABC be a triangle. Using only straightedge and compass, construct a point $P \in \overline{AB}$ and a point $Q \in \overline{BC}$ such that $\overline{AP} \cong \overline{PQ} \cong \overline{QC}$.

Problem 9.2

Source: [8, problem 16].

Prove that any two quadrilaterals are congruent if and only if their internal angles are congruent, in order, and their diagonals are congruent, in order.

Using only straightedge and compass, construct a quadrilateral, given its angles, in order and its diagonals, in order.

Problem 9.3

Source: [10, problem 14].

Prove that any two quadrilaterals are congruent if and only if their sides are congruent, in order, and the segments between the midpoints of their first and third sides are congruent.

Using only straightedge and compass, reconstruct a quadrilateral, given segments congruent to its four sides, in order, and a segment congruent to the segment between the midpoints of the first and third sides.

Problem 9.4

Source: [8, problem 19]; [10, problem 6]; [11, problem 3].

Given a point and an angle on a plane, construct, using only straightedge and compass, a line through the point that cuts the angle into a triangle of minimum perimeter.
Additionally, given also a segment, construct a line through the point that cuts the angle into a triangle whose perimeter is the length of the segment.

Problem 9.5

Source: [8, problem 22]; [10, problem 7]; [11, problem 4].

Given a circle and one of its diameters, and given a point on the plane that does not lie on the circle nor on the line containing the diameter, construct, using only a straightedge, the perpendicular from the given point to the given diameter.

Problem 9.6

Source: [21].

Given a circle and one of its diameters, and given a point on the circle, distinct from the endpoints of the diameter, construct, using only a straightedge, the perpendicular from the given point to the given diameter.

Problem 9.7

Source: [8, problem 23].

Given a segment and a positive integer n, divide the segment into n parts of equal lengths, using only a compass.

Problem 9.8

Source: [10, problem 16]; [11, problem 10].

Given two parallel segments and a positive integer n, divide one of the segments into n parts of equal lengths, using only a straightedge.

Problem 9.9

Source: [8, problem 39].

Determine for which $n, k \in \mathbb{Z}^+$ it is possible, given k segments of lengths $a_1, \dots, a_k$ on the plane, to construct, using only straightedge and compass, a segment of length b such that:

$$\sqrt[n]{b} = \sum_{i=1}^{k} \sqrt[n]{a_i}$$

Note: a segment of length 1 is not given.

Problem 9.10

Source: [10, problem 20]; [11, problem 13].

Reconstruct a square given one point from each side, using only straightedge and compass.
(Is such a square uniquely determined? Under what conditions? Maybe if the given points are not themselves vertices of a square?)
(What about a point on each of the lines containing the sides?)
(What about other polygons? What about solids in higher dimensions?)

Problem 9.11

Source: [14] and [23, problem 23]; [1, page 7, footnote 16].

Using only straightedge and compass, construct the directrix and focus of a given parabola.
(What about similar problem for other conics?)

Problem 9.12

Source: [7].

Using only straightedge and compass, construct the center of a given sphere.
(What about higher dimensions?)

Other

Problem 10.1

Source: [13, problem 6].

Let R, S, T be sets, with $\#(R) \geq 2$, $\#(S) \geq 2$, and $\#(T) \geq 3$. Let $f : R \times S \to T$ be such that $\#(\mathrm{im}(f)) \geq 3$. The elements $a \in R$ and $b \in S$ are such that the functions $S \to T : y \mapsto f(a, y)$ and $R \to T : x \mapsto f(x, b)$ are not constant.
Prove that there exist $p, r \in R$ and $q, s \in S$ such that $f(p, q)$, $f(r, q)$, $f(p, s)$ are all distinct.

Problem 10.2

Source: [6, pages 11–12, page 6 in the Russian version].

Let $f : \mathbb{R} \to \mathbb{R}$ be a function, with $f(0) = 0$, $f(1) = 1$, $f(88) = \sqrt{2}$. Prove that there exist $x, y \in \mathbb{R}$ with $|x - y| \leq 1$ such that $f(x + 1) > f(x)$ and $f(y + 2^n) \neq f(y)$ for some $n \in \mathbb{N}$.

References

[1] Aleksandr Sergeyevich Demidov. *On entrance examinations (conclusions and proposals based on analysis of documentary data)*. URL: https://www.lirmm.fr/~ashen/senderov/demidov.pdf. Archive: https://web.archive.org/web/20220121234744/https://www.lirmm.fr/~ashen/senderov/demidov.pdf.

[2] Jay Egenhoff. “Math as a tool of anti-semitism”. In: *The Mathematics Enthusiast* 11.3 (2014), pp. 649–664. DOI: 10.54870/1551-3440.1320. URL: https://scholarworks.umt.edu/cgi/viewcontent.cgi?article=1320&context=tme. Archive: https://web.archive.org/web/20240412101427/https://scholarworks.umt.edu/cgi/viewcontent.cgi?article=1320&context=tme.

[3] Grigory Abelevich Freiman. *It seems I am a Jew*. Southern Illinois University Press, 1980. URL: https://archive.org/details/freiman-english-it-seems-i-am-a-jew. Also available at https://www.lirmm.fr/~ashen/senderov/freiman-english.pdf (archive: https://web.archive.org/web/20220121234736/https://www.lirmm.fr/~ashen/senderov/freiman-english.pdf). Russian version (does not have the appendices): https://archive.org/details/freiman-rus-tex and https://www.lirmm.fr/~ashen/senderov/freiman-rus-tex.pdf (archive: https://web.archive.org/web/20220121234713/https://www.lirmm.fr/~ashen/senderov/freiman-rus-tex.pdf).

[4] Edward Vladimirovich Frenkel. *Love and Math: the heart of hidden reality*. Basic Books, 2013. ISBN: 978-0-465-06995-8. URL: https://www.edwardfrenkel.com/lovemath/. The relevant excerpt can be found at https://web.archive.org/web/20171010070710/https://www.newcriterion.com/issues/2012/10/the-fifth-problem-math-anti-semitism-in-the-soviet-union; Russian version: https://inosmi.ru/20121111/202009718.html (archive: https://web.archive.org/web/20260209065935/https://inosmi.ru/20121111/202009718.html).

[5] Boris Ilyich Kanevsky and Valery Anatolievich Senderov. “Admission results of graduates of six Moscow Schools to the Faculty of Mechanics and Mathematics at Moscow State University”. In: *Discrimination against Jews in University Admissions*. Ed. by Moscow Helsinki Group. 1979. URL: https://www.lirmm.fr/~ashen/senderov/mhg-112+pril.pdf. Archive: https://web.archive.org/web/20220121234719/https://www.lirmm.fr/~ashen/senderov/mhg-112+pril.pdf.

[6] Boris Ilyich Kanevsky and Valery Anatolievich Senderov. *Intellectual Genocide: entrance examinations for Jews at MGU, MFTI, MIFP*. 1980. DOI: 10.1142/9789812701169_0004. Included in [15, pages 129–152]. Russian version available at https://www.lirmm.fr/~ashen/senderov/ig-text.pdf (archive: https://web.archive.org/web/20250505152510/https://www.lirmm.fr/~ashen/senderov/ig-text.pdf); typewritten original (in Russian): https://www.lirmm.fr/~ashen/senderov/ig-color.djvu (archive: https://web.archive.org/web/20260201181835/https://www.lirmm.fr/~ashen/senderov/ig-color.djvu).

[7] Vladimir Kelman. 2013. URL: https://kerosinka.livejournal.com/1633.html?thread=383329#t383329. Archived at https://web.archive.org/web/20260222040201/https://kerosinka.livejournal.com/1633.html?replyto=383329, at https://archive.today/CF4qL#t383329, at https://ljsear.ch/savedcopy?comm=1194975920, and at https://web.archive.org/web/20260201215430/https://ljsear.ch/savedcopy?comm=1194975920.

[8] Tanya Khovanova. *Main list of math "coffin" problems*. 2005. URL: https://www.tanyakhovanova.com/Coffins/coffinsmain.html. Archive: https://web.archive.org/web/20251207003425/https://www.tanyakhovanova.com/Coffins/coffinsmain.html.

[9] Tanya Khovanova. *Math "coffin" submitted by others*. 2008. URL: https://www.tanyakhovanova.com/Coffins/othercoffins.html. Archive: https://web.archive.org/web/20250517004330/https://www.tanyakhovanova.com/Coffins/othercoffins.html.

[10] Tanya Khovanova and Alexey Radul. *Jewish Problems*. Preprint of [11]. 2011. DOI: 10.48550/arXiv.1110.1556. URL: https://arxiv.org/abs/1110.1556. Archive: https://web.archive.org/web/20260211015757/https://arxiv.org/pdf/1110.1556.

[11] Tanya Khovanova and Alexey Radul. "Killer Problems". In: *The American Mathematical Monthly* 119.10 (2012), pp. 815–823. DOI: 10.4169/amer.math.monthly.119.10.815.

[12] Moscow Helsinki Group. *Discrimination against Jews in University Admissions*. 1979. URL: https://www.lirmm.fr/~ashen/senderov/mhg-112+pril.pdf. Archive: https://web.archive.org/web/20220121234719/https://www.lirmm.fr/~ashen/senderov/mhg-112+pril.pdf.

[13] Melvyn Bernard Nathanson. "Appendix B". In: *It seems I am a Jew*. Southern Illinois University Press, 1979, pp. 95–96. URL: https://archive.org/details/freiman-english-it-seems-i-am-a-jew. Also available at https://www.lirmm.fr/~ashen/senderov/freiman-english.pdf (archive: https://web.archive.org/web/20220121234736/https://www.lirmm.fr/~ashen/senderov/freiman-english.pdf). Russian version (does not have the appendices): https://archive.org/details/freiman-rus-tex and https://www.lirmm.fr/~ashen/senderov/freiman-rus-tex.pdf (archive: https://web.archive.org/web/20220121234713/https://www.lirmm.fr/~ashen/senderov/freiman-rus-tex.pdf).

[14] Alexander Khaniyevich Shen. "Entrance examinations to the Mekh-mat". In: *The Mathematical Intelligencer* 16.4 (1994), pp. 6–10. DOI: 10.1142/9789812701169_0008. URL: http://www.3038.org/press/shen.pdf. Archive: https://web.archive.org/web/20260107193055/http://www.3038.org/press/shen.pdf. The text in Russian language has more information and contains more problems; it is available at https://www.lirmm.fr/~ashen/alexander-shen.narod.ru/vershik.pdf (archive: https://web.archive.org/web/20240918040436/https://www.lirmm.fr/~ashen/alexander-shen.narod.ru/vershik.pdf) and at https://alexander-shen.narod.ru/vershik.pdf (archive: https://web.archive.org/web/20250912070826/https://alexander-shen.narod.ru/vershik.pdf).

[15] Mikhail Arkadyevich Shifman, ed. *You failed your math test, Comrade Einstein. Adventures and misadventures of young mathematicians, or test your skills in almost recreational Mathematics.* World Scientific, 2005. ISBN: 9789812563583. DOI: 10.1142/5791. URL: https://www-users.cse.umn.edu/~shifman/EinsteinBook.pdf. Archive: https://web.archive.org/web/20251004025002/https://www-users.cse.umn.edu/~shifman/EinsteinBook.pdf.

[16] Mikhail Arkadyevich Shifman. *Through the prizm of time — Angles of reflection.* Epilogue to "Comrade Einstein". URL: https://www-users.cse.umn.edu/~shifman/Epilogue_12_28_2016.pdf. Archive: https://web.archive.org/web/20251116145012/https://www-users.cse.umn.edu/~shifman/Epilogue_12_28_2016.pdf. A version updated to 2020 is available at https://www.lirmm.fr/~ashen/senderov/shifman2020.pdf (archive: https://web.archive.org/web/20220121234737/https://www.lirmm.fr/~ashen/senderov/shifman2020.pdf); another updated version, differing only in the last two pages, is available at https://www.academia.edu/101548394/.

[17] Unknown. *Selected entrance examination problems at the Faculty of Mechanics and Mathematics of Moscow State University in 1979.* URL: https://www.lirmm.fr/~ashen/senderov/zadachi-1979-color.djvu. Archive: https://web.archive.org/web/20260227111932/https://www.lirmm.fr/~ashen/senderov/zadachi-1979-color.djvu.

[18] Unknown. *Selected entrance examination problems at the Faculty of Mechanics and Mathematics of Moscow State University in 1982.* URL: https://www.lirmm.fr/~ashen/senderov/zadachi-1982.djvu. Archive: https://web.archive.org/web/20260223071158/https://www.lirmm.fr/~ashen/senderov/zadachi-1982.djvu.

[19] Unknown. *Selected entrance examination problems at the Faculty of Mechanics and Mathematics of Moscow State University in 1986.* URL: https://www.lirmm.fr/~ashen/senderov/zadachi-1986-color.djvu. Archive: https://web.archive.org/web/20260226060451/https://www.lirmm.fr/~ashen/senderov/zadachi-1986-color.djvu.

[20] User "arbat" on LiveJournal. 2003. URL: https://bbb.livejournal.com/787297.html?thread=2688609#t2688609. Archived at https://ljsear.ch/savedcopy?comm=1344985597, at https://web.archive.org/web/20260201212300/https://ljsear.ch/savedcopy?comm=1344985597, and at https://archive.today/OX7Ks. See also https://bbb.livejournal.com/787297.html?thread=2698081#t2698081, archived at https://ljsear.ch/savedcopy?comm=1875114955 and at https://web.archive.org/web/20260201212808/https://ljsear.ch/savedcopy?comm=1875114955); see also https://bbb.livejournal.com/787297.html?thread=2708577#t2708577, archived at https://ljsear.ch/savedcopy?comm=1236687776 and at https://web.archive.org/web/20260201212439/https://ljsear.ch/savedcopy?comm=1236687776; see also https://bbb.livejournal.com/787297.html?thread=2733921#t2733921, archived at https://web.archive.org/web/20260211051731/https://bbb.livejournal.com/787297.html?replyto=2733921 and at https://archive.today/z2tE0#t2733921.

[21] User “azb1958” on LiveJournal. 2013. URL: https://azb1958.livejournal.com/34112.html%5D(https://azb1958.livejournal.com/34112.html. Archive: https://web.archive.org/web/20210925084745/https://azb1958.livejournal.com/34112.html; see also https://azb1958.livejournal.com/34112.html?thread=141376#t141376, archived at https://web.archive.org/web/20260307191841/https://azb1958.livejournal.com/34112.html?replyto=141376.

[22] User “nieuwe_zijde” on Blogger. 2009. URL: https://avzel.blogspot.com/2009/07/blog-post_30.html?showComment=1254948495163#c1386969495227008652. Archive: https://web.archive.org/web/20190610091539/https://avzel.blogspot.com/2009/07/blog-post_30.html?showComment=1254948495163#c1386969495227008652.

[23] Ilan Vardi. *Mekh-Mat entrance examinations problems*. The final version of the article is contained in [15, pages 22–95]. 2000. DOI: 10.1142/9789812701169_0001. URL: https://www.lix.polytechnique.fr/Labo/Ilan.Vardi/mekh-mat.ps. Archive: https://web.archive.org/web/20251225204859/https://www.lix.polytechnique.fr/Labo/Ilan.Vardi/mekh-mat.ps. PDF version: https://www.tanyakhovanova.com/Coffins/Vardi-solutions.pdf (archive: https://web.archive.org/web/20251207000147/https://www.tanyakhovanova.com/Coffins/Vardi-solutions.pdf).